amplification of mathematical theory applicable to biological problems—important as this unquestionably is. The pressing need is for biologists themselves to apply more extensively, intelligently, and critically, theory which has already been made available to them.

History seems to demonstrate that mathematics can not be introduced into any science by mandate. It must take its place in the physical, biological and social sciences through the normal evolutionary development of these sciences themselves.

Physical and biological sciences have passed through, or are now in, periods of observation and description, of classification, of comparison, and of experimentation. Apparently all the physical and biological sciences move in their development toward a final period, not yet surpassed in any of the sciences, of mathematical description and analysis, and of the formulation of mathematical laws. It is in part because of an earlier adoption of mathematical notation and methods of reasoning that in their development physics and chemistry shot ahead of biology. It is because of the difficulty of obtaining adequate quantitative data and because of the unwillingness of many of the workers to employ quantitative methods of analysis that sociology lags far behind biology.

The evolution of physics and of chemistry to the stage of mathematical sciences has been much more rapid than has that of biology. In exoneration of the biologist, however, we must remember that the amount of descriptive work which he has had to do is enormously greater than that of the chemist or physicist, and that the problems of structure and function with which he has to deal are vastly more complex and difficult.

While most biologists were long hostile to the introduction of mathematical methods into biological research, it was inevitable that in time they would find it necessary to avail themselves of the powerful tools of mathematical theory.[1]

As I have indicated in an earlier address, mathematics has, during the past quarter of a century, been entering biology on two broad fronts.

First, biology in our generation has become an experimental science. This, in itself, represents an evolutionary advance over its purely descriptive and comparative stages. Biology entered the experimental period when physics and chemistry were already far advanced as experimental sciences. It was, therefore, only natural that the experimentation of biologists should consist very largely in the introduction into biology of the methods of the more highly developed sciences. Controlled experimentation is essentially quantitative in both methods and results. Biology has, of necessity, taken over with the experimental methods of physics and chemistry the mathematical methods of description and analysis which alone are capable of dealing with quantitatively

[1] The progress in this field is a stimulating chapter in the history of science, which cannot be given here, even in outline.

measured variables and which, in consequence, early became an integral part of these two highly developed physical sciences.

Second, the work of Quetelet, Galton and Pearson, and of others who have been associated with Pearson in the movement which has often been designated as the Biometric School, has opened up to mathematical treatment a vast group of problems. These pertain to the description and analysis of phenomena which are highly variable and which can not, in many cases, be experimentally controlled.

Our time is brief, and since no special plea is required for the mathematics which form a part of physics and chemistry as independent sciences, or of physics and chemistry as they are a part of biology, I shall limit my discussion to the needs of biology for mathematics as it may be applied in the second group of problems. In relation to the first line of advance, I will merely say in passing that, in my opinion, we shall soon reach a stage in biology at which the type of mathematics which has served physics and chemistry will be inadequate, and at which experimental results obtained under controlled conditions will themselves have to be subject to the statistical method of treatment.

II. *The Specific Mathematical Requirements of Biology*

Because the biologist deals with phenomena which are all but infinitely complex, his demands upon the mathematician may ultimately be many in number and varied in kind. The specific needs which can now be recognized must be presented in barest outline. They may be grouped under four main heads.

1. The Need for Quantitative Description

Those groups of plant and animal organisms which are sufficiently different that they can be assigned to the categories which the biologist calls species may be numbered by hundreds of thousands. Each of these groups has its own unknown evolutionary history and its own range of tolerable environmental conditions. Experimental work indicates that the great majority of these so-called species are not homogeneous entities but heterogeneous groups of sub-species, minor species, varieties, or strains—whatever one cares to call them—each of which shows some constancy of character from generation to generation. The individuals of a species may be subject during the course of their development to (and may therefore have their characteristics modified by) endless permutations of those extrinsic factors which constitute the environment. Thus the first need of biology was observation and description of the actual living things which now inhabit, and have in the past inhabited, the planet. Centuries of effort have been given to qualitative work in this field.

As biological research progresses, description becomes more, not less, essential, but the description must be to an increasing extent in terms of mathematical constants and equations. Both biologists and mathematicians have sometimes failed to realize that many of the statistical constants are, in and of themselves, simply descriptive.

While such quantitative description has in the past been largely devoted to the physical and physiological characteristics of individuals at one and the same (generally the mature) stage of development, it is evident that the course of the development of the individual and the distribution of organisms through time and space must also be described in terms of mathematical expressions. Limiting our attention for the moment to the quantitative description of masses of individuals, we may review briefly some of the accomplishments which have been made and suggest some of the needs which must still be met.

a. The Description of Type and of Deviation from Type

Suppose N variates, $x_1, x_2, x_3, \cdots x_n$, all recognized by the trained biologist as belonging to the same category, to have been measured. If all individual values of x were identical, it would be unnecessary to determine even a simple mathematical expression to represent them. The value required would be obvious by inspection. Since the values of x differ among themselves, so that the characteristics of no one of them can be accepted as a wholly valid basis of generalization, it is necessary to obtain some description of the series as a whole based on a large number of individuals.

While such values as the most frequent or modal class, the maximum and the minimum class, were widely used in earlier semiquantitative biological work, they are inadequate since they neglect a material (and often a large) fraction of the available measurements. $\bar{x}=\Sigma(x)/N$ furnishes a description of the magnitude of x based upon all of the individuals considered. If the values of x have been accurately measured, $\bar{x}$ is, within the limitations evident in the formula, a true description of the magnitudes of the individuals actually measured.

The description of x in terms of $\bar{x}$ is inadequate, since by its very nature the average obliterates certain of the most salient features of the series of variates. For biological comprehensiveness the variability of x must also be expressed. The biologist has learned to do this in terms of the "standard deviation" or "root mean square deviation."

$$\sigma_x = (\Sigma[(x - \bar{x})^2]/N)^{1/2}$$

Such quantitative descriptions of the characteristics of series of variates as $\bar{x}$, $\bar{y}$, etc., are directly comparable for samples within the same category. The values for different categories may often be rendered comparable by expressing $\bar{x}$, $\bar{y}$, etc., as ratios to some properly chosen standard values, $\overline{X}$, $\overline{Y}$. σ_x, σ_y, etc. may be directly compared, or compared after reference to their mean values, $V_x=\sigma_x/\bar{x}$, $V_y=\sigma_y/\bar{y}$, etc., in order to correct for differences in the average magnitudes of the variables.

While such measures of type and variability are among the simplest of the mathematical descriptions of the characteristics of groups of organisms, there are as yet largely undeveloped possibilities in biological research through the proper use of these constants. The problems are primarily biological.

Research for the most part requires only the use by the biologist of formulae which are already available.

It will be clear that while only the biologist can make the work of the mathematician fruitful of valuable results through the application of his constants, only the mathematician can justify the use of the root of the average squared deviation of the variates from their mean as the most practicable and generally usable measure of the amount of variability. In many cases the root mean square deviation is altogether inadequate as a full description of the biological facts. In some instances I believe it may be quite erroneously applied. Certainly the latter is true of the coefficient of variation, $V_x = \sigma_x / \bar{x}$.

Furthermore there are still undeveloped possibilities in the theory underlying these descriptive constants. One such field is the combination of the ordinary biometric theory with the theory of measurements. In biological work it is ordinarily assumed that the individual measurements represent the true values of the variates, and that the trustworthiness of the constants derived from N such values is determined solely by the magnitude of the errors of random sampling. This assumption is valid in only a very approximate way. As work of greater precision is undertaken in biology it will be necessary to take errors of measurement into account in the interpretation of results.

Finally, there are always dangers associated with methods which are so simple that they can be used in a wholly routine way. Even at the present stage of development of theory and application, it is important to keep clear of dogmatism. On the one hand, the mathematician should not be too positive in his assertion that such constants as means, standard deviations, and coefficients of variation are the best descriptions of the actual observations, because he does not know all of the special needs of biological work. On the other hand, the biologist should not be too ready to apply blindly what the mathematician may recommend. He should know something of the reasons for the recommendation given, and should assure himself that the reasons are wholly cogent for the specific case with which he has to deal.

b. The Description of Frequency Distributions

Such constants as $\bar{x}$, σ_x, V_x convey vastly more and more valuable and dependable information concerning a distribution in which frequencies are plotted as ordinates against the values of x as abscissae than any other kind of description (*e. g.*, the empirical mode and the observed minimum, maximum and range). These fundamental constants are, however, inadequate as a full portrayal of the distribution of the magnitudes of x. The frequencies may be distributed symmetrically about the mean, may be wholly skew, or may take any other conceivable form. Some description of the frequency distribution as a whole is therefore essential.

The biologist can represent empirical frequencies graphically, and can compare two or more distributions of the values of different variables or of the same variable observed under different conditions. Before much progress

can be made, however, the empirical distributions must be described or graduated by some kind of frequency curve equations. Quite obviously the mathematician and not the biologist must provide suitable equations.

Early in the history of the description of the frequency distributions of biological measurements it was assumed that in general such variates are distributed in accordance with the Gauss-Laplace or normal curve. We now know that this holds rigidly in only a fraction of the cases. The form of the frequency distribution of x, as well as its average magnitude and standard deviation, may be characteristic of the particular variable under consideration, and may be of great significance in the analysis of biological phenomena.

Two courses are open to the mathematician in attempting to describe such a distribution by an equation. *First*, he may try to obtain an equation which will give exactly the observed frequencies of x. In general this will be exceedingly laborious, and in most cases practically impossible. *Second*, he may seek to represent the results as well as they can be represented by an equation involving moment coefficients of a lower order than those required for the more accurate description of the actual facts of observation.

The proper solution of this problem is of more interest to the biologist than to the pure mathematician. The mathematician may readily provide symbolic expressions of any degree of complexity. The practicability of applying these to biological data, and the extent to which they will be used in actual biological research, will be determined very largely by the arithmetical routine required in the numerical evaluation of the equations.

Furthermore, there is a limit beyond which precision of mathematical description of biological phenomena ceases to be of biological value. The results of biological observation and experimentation are irregular because of the inevitable errors of measurement and because of the errors of random sampling from a population made up of individuals which are highly variable because they have been subjected during their development to the influence of the permutations of an unknown but generally large number of innate or extrinsic factors, which themselves may be of highly varying potency.

If the mathematician followed the first course indicated above to the limit, he would merely reproduce by his equations the very irregularities which the biologist would like to see eliminated. He would leave in the archives of biology a statement which equals in complexity and irregularity that of the original biological data upon which it was based. If, however, he adopts the second alternative, he does not reproduce, but graduates, the frequencies. The biologist requires the simplest description which is adequate to represent his observations reasonably well, that is, well enough for the purposes of biological science as it exists today. It must be clearly recognized, however, that at any time more refined, and in consequence more complex, mathematical formulae may be required and may be justified by the progress in exactness of biological research.

Large contributions of method have been made by mathematicians. As

mere descriptions of individual frequency distributions, their equations may be adequate. The work of description in such terms has not, however, been sufficiently extensively and systematically pursued by biologists to have made the work very fruitful of purely biological results. It is doubtful whether this end will be realized until biologists as a class have much more thorough general mathematical training than is possible at present.

Furthermore, biological phenomena as they appear on the surface may not be interpreted by description, however accurate. A theoretical curve which gives frequencies in terms of areas corresponding to given intervals on the scale of abscissae may be erroneous because the units of the latter scale, while numerically comparable, have not the same biological significance. There is here a practically untouched field awaiting the attention of the biologist with an adequate mathematical background.

Another group of important problems in the mathematical description of biological data must be passed with but a word of comment. Frequency curves, which describe the form of the distribution of magnitudes of biological variables at any given stage of development or under any given set of conditions, fail to take account (except in the terms of end results) of the fact that organisms acquire their magnitudes at the given stage through the developmental processes which the biologist summarizes by the term *ontogeny*.

Comparative and experimental embryology have vastly increased our knowledge of ontogeny, but development is a quantitative process and to be fully understood must be dealt with, in some of its phases at least, in terms of mathematical descriptions.

A flood of papers has in recent years been devoted to the mathematical description of growth—both of the individual and of populations of individuals. Much of this work is superficial in the extreme, but sufficient results of value have been obtained to indicate the importance of the development of the field from both the mathematical and the biological sides.

c. The Quantitative Description of Inter-relationship

The biologist cannot be satisfied with the exact description of single biological variables. He cannot limit his attention to x alone but must also consider the relation between x and any number of other variables, $a, b, c, \ldots, \alpha, \beta, \gamma, \ldots$. He knows that in nature the magnitudes of x and of these other variables are not independent but that they are interdependent because x may be causally related to one or more of the other variables, or because x and the other variables may be subject in a similar manner but in varying degrees to the influence of the same constellation of factors which determine their magnitudes.

In conventional terms, the biologist is interested in the problem of correlation. He requires to know, in the quantitative terms of a universally comparable scale, the closeness of interrelationship between any pair of variables.

The correlation coefficient—the brilliant conception of Francis Galton's

fertile mind—has long been familiar in biology. It was soon adopted in psychology and economics, and it is now finding its way into all of the sciences, including a field already so highly developed mathematically as astronomy.

Pearson early expressed Galton's conception of correlation in terms of

$$r_{xy} = \frac{\Sigma[(x - \bar{x})(y - \bar{y})]/N}{\sigma_x \sigma_y}.$$

Since the numerator of the right hand side involves the product moment of x and y about their respective means as origins, both x and y must be quantitatively measurable. When x and y are wholly independent and N is large the product tends to vanish, and r_{xy} to approach 0. Since the deviations of the variates from their respective means are expressed in terms of $(x-\bar{x})/\sigma_x$, $(y-\bar{y})/\sigma_y$, $r_{xy}=1$ when the relationship between the two variables is linear and when there is no variability in the arrays of one variable associated with the several classes of the others. If the deviations of both x and y from their respective means are preponderantly of like sign when both magnitude and frequency are considered, the value of r_{xy} will be positive. If deviations of pairs of x and y from their respective mean values are preponderantly of unlike sign, the value of r_{xy} will be negative. Interrelationships between two variables both measured on a quantitative scale and linear in their relation to each other are ideal for the biologist. Fortunately the experience of the past forty years has shown that the majority of relationships between morphological variables may be approximately expressed by straight line equations.

In the extension of the conception of correlation beyond the first simple case, three main difficulties have had to be faced. Possibly there are others which have not yet been so clearly distinguished.

First, the relationship between x and y may be non-linear. In this case r_{xy} does not furnish a wholly valid description of the relationship between them.

Second, some characteristics of organisms cannot as yet be measured on a strictly quantitative scale, but must be expressed as qualitative categories. Thus the coat color of cattle or the hair or eye color of human beings may be more readily described than measured.

This introduces all the difficulties associated with the real or imaginary discontinuity of variation of many biological variables. Whether or not really discontinuous, a large percentage of biological variables must be expressed on a discontinuous scale. This introduces theoretical questions as to the applicability to discontinuous phenomena of mathematical theory based on the assumption of continuous variation. Even if these theoretical difficulties be ignored, or considered of insignificant importance in dealing with biological phenomena, there still remain in some cases large practical obstacles which may be more readily considered later.

Third, the number of classes which can be distinguished in the case of measurable characters, and particularly the number of categories which may be recognized in the case of non-measurable characters or attributes, may be

either large or small. Thus individuals may fall for purposes of registration statistics into as few as two alternative groups—for example, survived and died, or vaccinated and not vaccinated, or married and unmarried. When the number of classes of one or both of the two variables is either too small or too large, difficulties may be encountered in the measurement of the interrelationship between them. In some cases it is possible to increase the number of categories by logical subdivision, when too small, or to decrease the number by logical combination, when too great. In other instances the data available to the mathematical biologist are not such as to make this possible. Even in the most careful biological research it is sometimes practically necessary to recognize two alternative categories only, or to retain a number of categories so great as to render the interpretation of contingency coefficients uncertain.

Limiting our attention for the moment to the third difficulty of biological origin, with a promise to return to the first and second classes later, we may note that in biological research, when conducted in a broad and comparative way, we have for either of our two variables, x and y, all possible gradations between measures on a uniformly divided quantitative scale, numerous multiple categories and two alternative categories. In the determination of measures of interrelationship these may be encountered in every possible permutation.

It is for this reason that the theory of correlation has had to be extended to include the description in quantitative terms of interrelations between values of x and y measured or appraised in multiple categories only, and in various combinations of quantitative and non-quantitative categories. Thus we have the following possible combination of our two variables.

First Variable x (or y)	Second Variable y (or x)
Quantitatively measured: classes unlimited in number	Quantitatively measurable; classes unlimited in number
"	Quantitatively measurable; classes limited in number
"	Classed in multiple categories; classes many
"	Classed in multiple categories; categories few or two only
Classed in multiple categories: categories many	Classed in multiple categories; categories many
"	Classed in multiple categories; categories few or two only
Classed in multiple categories; categories few or two only	Classed in multiple categories; categories few or two only

All of these interrelationships have been considered in a series of masterly papers by Pearson and others who have followed him. Formulae have been deduced which have in common the expression of the interrelationships in terms of a scale which is universally comparable in that it has a range of 0 to 1. Ideally, the point 0 on this scale represents an entire absence of interrelation-

ship between the two variables, while ideally $+1$ or -1 indicates such closeness of interrelationship that knowing the value of one variable, x, we also know the value of the other variable, y, of the same pair.

This description of the interrelationship between two variables on a quantitative, universally comparable, and mentally comprehensible scale has been one of the most fertile conceptions in biology. A volume would be required to list the applications which have already been made.

Illustration of the importance of biological accomplishments already realized by the application of available correlation and contingency theory is less pertinent to our present purposes than the indication of some of the difficulties which still remain to be overcome. Of fundamental importance in relation to the problem of biological methodology is the fact that the description of the relationships between biological variables as written in the quantitative terms of the various interrelationship coefficients are not always wholly consistent.

Consistency of the end results of mathematical reasoning has long been recognized as one of the touchstones by which the validity of the premises and the deductions from them are tried. Inconsistency in quantitative descriptions of interrelationships may be due either to the inadequacy of the theory as far as it is at present developed and tested, or it may be due to the fact that the two quantitative descriptions really portray different features of the biological interrelationships.

Some of the difficulties are such that they will long remain insurmountable, but both biologists and mathematicians should keep before them the fact that the needs for the adequate description of the relationship between the variables measured or appraised for N individuals or associated pairs of individuals has been only partly met.

Returning now to the first and second difficulties we may note the following.

While the measurement of interrelationship between variables in the terms of a universally comparable scale has been one of the most fruitful conceptions in modern science, measures on a universally comparable scale must fail to express the relationship between two variables in terms which are the most useful for purposes of many biological researches. It is for this reason that measures of interrelationship in terms of correlation must also be expressed in terms of regression equations. These equations may be of various orders. Most fortunately for biological research, experience has shown that the linear equation

$$(y - \bar{y}) = r_{xy}\frac{\sigma_y}{\sigma_x}(x - \bar{x})$$

is realized to a very close degree of approximation for an enormous number of pairs of variables. When regression is not linear, the theory of the correlation ratio involving ${}_x\eta_y$, ${}_y\eta_x$ must be employed to lead to equations of degrees higher than the first.

In the use of the correlation ratio as a method of approach to non-linear regression equations, a number of difficulties are incurred. While some progress has been made toward overcoming these, much still remains to be done in the development of mathematical theory and in testing it through its application in actual biological research.

In the case of characters or attributes which are not quantitatively measurable, we must note that it is not yet possible to deal with many of these in terms of regression at all. In some cases, indeed, all that is possible is to express the deviation of the system from independent probability in terms of the correlation scale. Much must be done by both mathematicians and biologists before we shall be able to deal satisfactorily with all the problems of the interrelationship between variables of this kind regularly encountered in biological research.

In the foregoing discussion, we have dealt with the problems of the interrelationship between pairs of variables, x and y. In certain cases we have, instead of individual pairs, a number of values of x or y or of x and y which are associated in groups or classes, which groups or classes may be delimited one from the other by reason of their position in time or space, or by some other valid condition. Such groups of x, or of any other variable, may be designated as a class.

Now all of the n values of x which constitute one of the m classes which together make up the $\Sigma(n) = N$ individuals of the sample may be assumed to be subject to the influence of a constellation of intrinsic or extrinsic factors in some measure peculiar to that class and competent to determine or to modify their values. In short, the biologist frequently has to deal with variables the magnitudes of which may be associated with or differentiated by time or space, or by both time and space, or by some other differentiating factor which is not covered by these terms as they are ordinarily employed.

Because of these conditions, there will be a tendency for the individuals of the several classes to be differentiated from class to class. This differentiation may be measured in terms of the intra-class correlation coefficient. Such intra-class correlation coefficients may be computed for values of x or of y or of any other variable. Not infrequently values of both x and y, or of more variables, are measured on individuals of the same class, or on individuals of associated classes. In such cases it is possible to deal with direct or cross intra-class correlations, or with direct or cross inter-class correlations. In some instances it may be desirable to recognize sub-classes, in which case fractional intra-class or inter-class correlation coefficients may be determined.

Large biological accomplishments await the further development and application of intra-class and inter-class correlation theory.

Two cases in point may be noted. *First*, in many instances the influence of the factors associated with time and space on the values of a variable must be measured in terms of the correlations between the magnitudes of the associated variates themselves, because the constellation of influences associated

with time or space is so complex that the worker has no means of determining directly the correlation between these environmental factors and the magnitudes of the variable. *Second*, if the biologist is to determine the true values of the variables with which he has to deal, he must eliminate the differentiating influence of time or space or of both time and space on the empirical measurements which he has made.

The foregoing discussion has been limited to the measurement of the interrelationship between two variables. Cases in which more than two variables must be considered at one time will be more conveniently treated later.

2. The Need for Criteria of the Adequacy of Mathematical Description

Having, as a biologist, urged that the equations to be employed in the description of organic phenomena should be deduced by processes approved by the trained mathematician, I must, as a biologist, insist that such mathematical descriptions or equations should be satisfactory to the biologist.

If mathematical descriptions of biological phenomena are to be written, we require rigorous mathematical tests of the closeness of agreement between the empirical and the theoretical distributions. We need to know, in short, whether the equation adopted describes the actual biological facts so closely that differences between the empirical values and the theoretical values as given by the equation may be no larger than those which might reasonably be supposed to have arisen from errors of measurement or from errors of random sampling.

Thus the biologist requires not merely a system of mathematical equations which will describe his observational or experimental data, but he must be assured that the agreement between observational data and mathematical description is as good as can be expected from the random sampling of a population of the given variables, or at least that it is good enough for the purposes of his investigation. If the results do not show this concordance, then he may properly be dissatisfied with the given equation as a basis of generalization concerning his measurements, and he must seek some other equation to represent his empirical results.

It will be clear that in addition to providing the equations for mathematical description of series of biological data, the mathematician has a two-fold task. *First*, he must provide criteria of the closeness of agreement of his mathematical description with the emprical data of observation or experimentation. *Second*, he must provide a system of probable errors which determine the trustworthiness of his expressions as bases of generalization. Naturally the processes underlying these two undertakings are interrelated.

Limiting our attention for the moment to the first of these two requirements, we may merely note the importance of Pearson's χ^2 and P criterion of goodness of fit, more recently extended to the testing of the sameness of origin of two independent samples, and of Pearson's and Blakeman's work on criteria of linearity of regression. These tests of the adequacy of mathematical description

have been so fruitful that it is earnestly to be hoped various and much needed extensions will be made along different lines.

3. The Need for Criteria of Bases of Generalization

The purpose of biological work is to derive from the series of observations which can be made and recorded as descriptions of relatively small samples of biological variables, systems of generalizations concerning the biological phenomena of the universe.

As descriptions of the N observations or sets of observations made on any given biological variable, the physical constants (such as means, standard deviations, coefficients of variation, coefficients of correlation, regression equations, etc.) are absolutely valid except for (a) the limitations of the errors of measurement (which in some cases vanish on summation) and for (b) the limitations due to the assumptions made in the selection of the mathematical formulae employed.

Such validity is not, however, all that is required to fulfill the purpose of biology as indicated above. The precise determination of the characteristics of the individuals of a specific group, and the adequate description of the group in the terms of biometric constants, may have large personal interest or economic importance but the procedure lacks scientific value unless it can be made the basis for a generalization concerning the universe or population of individuals from which the finite sample investigated was drawn. Since, because of practical limitations, the whole of the population cannot be investigated, the characteristics of its individuals must be predicted from the sample. The mathematician must provide the biologist with the means of estimating how trustworthy his statistical constants are as a basis for generalization concerning the population, or the universe.

The validity as bases for generalization of the mathematical constants which describe series of biological measurements must depend upon four conditions:

(1) The method of drawing the sample must introduce no differentiation between the sample and the population other than that reasonably attributable to the errors of random sampling.

(2) The measurements must be made with a degree of precision sufficient for the purposes of the generalization to be drawn.

(3) The formulae in terms of which the final constants for the samples are expressed must introduce no source of error.

(4) The number of individuals measured must be adequately large for the generalization to be formulated.

Absolute assurance that the fraction which has been subjected to observation or measurement is really typical of the universe as a whole, as required by condition (1), is unattainable except by the process of determining the characteristics of all the individuals which constitute the universe or the population at large. Because of the impracticability of measuring all of the individuals of the population, the mathematician has quite properly laid down the principle

that the sample observed must be drawn at random. This is relatively easy when the population sampled is made up of colored balls or of dice contained in the mathematicians bag of infinite capacity.

Living organisms are not, however, found under these ideal conditions. They are not merely distributed through time and space, but their distribution is very definitely related to the differences in the conditions presented by time and space. A sample taken at one place or time may as a matter of fact be highly differentiated from other samples which might be taken. Only biological experience, with a consideration of all of the possible physical and biological factors involved, can make reasonably certain the adequacy of the sampling of the population in the case of any unique sample.

When more than a single sample is taken, the constants obtained from the several samples may be compared among themselves. In this case the mathematician can furnish criteria of the randomness with which the samples were taken. In my opinion there should be further provision of theory for dealing with the problem of the significance of the results of repeated sampling of what is presumably the same population.

In some cases biologists have followed the mathematician so blindly as to presume that a sample distributed at uniform intervals throughout the time or space occupied by a fraction of the population is less suitable than a sample drawn in a wholly haphazard manner. As a matter of fact, it often turns out that samples taken in a systematic manner are more representative of the population, and in consequence more valuable as a basis of generalization, than samples taken in what is supposed to be a random manner. This is true in part because (a) it assures the wider distribution of the individuals through time or space, and in consequence makes more probable the obtaining of a truly representative sample, and in part because (b) the individuals of the universe may be subject to the influence of differentiating factors associated with space or with time or with both space and time. Thus when the problem in hand is the comparison of two or more series of measurements (as is frequently or generally true in practical work in quantitative biology) it is possible by selecting pairs of individuals, x and y, which are distributed in a systematic manner through time or space, or through both, to obtain a smaller value of the probable error of the difference between them than would otherwise be possible. This is true because any positive correlation which may exist between x and y will tend to reduce the probable error of the difference between them.

The problem of accuracy of measurement (2) need not receive consideration here further than to note that it must be solved by the cooperation of biologist and mathematician. In most of the statistical theory as developed up to the present time, the assumption has been made that the measurements which are available for mathematical analysis represent the true values of the variables. In many cases this is far from true. The measurements themselves are in some degree erroneous. It will be necessary, therefore, in the future to combine the theory of measurement and the theory of variation to a greater extent than has heretofore been done in biological investigation.

Quite obviously only the mathematician can furnish the biologist assurance that condition (3) has been adequately met.

The mathematician must not, however, assume that this is a simple task. In biology constants to be of the greatest value must be comparable from investigation to investigation. The materials considered, the conditions under which the measurements were made or the units in which the measurements are recorded may vary greatly from case to case.

In consequence there may be a tendency on the part of the worker to use different methods of statistical analysis for the different series. This may, indeed, be necessary.

Consider by way of illustration the problem of the measurement of the relationship between the variables a and b when each may be classed in alternative categories only, a_1, a_2, b_1, b_2. Obviously the four-fold table is

	b_1	b_2	Total
a_1	$a_1\ b_1$	$a_1\ b_2$	$a_1b_1 + a_1b_2$
a_2	$a_2\ b_1$	$a_2\ b_2$	$a_2b_1 + a_2b_2$
Total	$a_1b_1 + a_2b_1$	$a_1b_2 + a_2b_2$	N

Now many methods of measuring the relationship between a and b under such conditions have been suggested, but certain of these methods may give very inconsistent results when applied under varying conditions. Clearly the mathematical theory underlying some of the formulae must be inadequate. The illustration is by no means unique. A vast amount of work still remains for mathematicians and biologists in the selection of the formulae most suitable for application in special cases.

Finally, with reference to (4), only the mathematician can decide how large a number of individuals must be measured in order to arrive at a generalization of any given degree of accuracy, and he can give the information required in any specific case only when he knows the experience of the biologist with respect to the variability of the character, or of the statistical constants for the character, in question.

Taking the simplest of the constants, such as $\bar{x}$ and σ_x, by way of first illustration of the difficulties of the problem, we may note that while $\bar{x}$ and σ_x are true descriptions, within limitations already indicated, of the sample of N individuals studied, they may be in some measure erroneous as descriptions of the population from which the values of x were actually drawn. Thus they have no probable error (except that due to random or non-systematic errors of measurement which do not vanish in the summation of the second and other even powers of the technical errors) as a description of the magnitude or the variability of the sample on which they are based, but are subject to the limitations of a probable error as a generalization concerning the population from which the finite (and

often small) sample was drawn. It is for this reason that we write $\bar{x} \pm E_{\bar{x}}$, $\sigma_x + E\sigma_x$.

While the theory of the probable errors of means and standard deviations is relatively easy as compared with that of the probable errors of constants involving higher moments, product moments, or products of independent probabilities, it is in itself by no means simple in either theory or practice. Thus, for example, the conventional formula, $E\sigma = .6754\sigma_x/(2N)^{1/2}$, usually but not invariably given in the text books, is valid only if the magnitude of the variable be distributed with normal frequency. To determine a value of $E\sigma$ which shall be correct for any distribution (and here neglecting the errors of generalization which may be due to either the size of N or the magnitudes of the standard deviation), the first four moment coefficients must be computed. When the probable errors are calculated by the two methods, there may be a difference of as much as 100 percent between them.

It must not be forgotten that in biological work the interpretation of the probable error involves the fundamental problem of the variability of the individuals of the sample and of the infinite population. If s_x be the standard deviation in the infinite population and σ_x the standard deviation of the N individuals of the sample, it is by no means certain that $\sigma_x = s_x$.

Since the standard deviation of the individuals of the infinite population s_x is unknown, the standard deviation of the individuals of the sample must be substituted as an approximate value. Both $\bar{x}$ and σ_x may be materially in error, expecially when N is small and s_x is large.

Furthermore, in many studies in practical biological or sociological work we deal with cases in which the population is not infinitely large. When N is a substantial part of the universe, the value of a given constant as a basis of generalization is somewhat different from that which it is when N is an infinitesimal part of the general population of which it is a random sample.

These points might be developed almost indefinitely. Quite obviously all statistical constants, including the many coefficients of correlation and contingency, have their probable errors. The deduction of the most suitable formulae for the probable errors is a task requiring mathematical ability and training of the highest order. An enormous contribution has been made to the field by Karl Pearson and his pupils, but much still remains to be done on the problem, which is one of the most important in the whole field of quantitative science.

4. The Need for the Powers of Symbolic Analysis

In the preceding sections it has been made sufficiently clear that the mathematician must, by processes of symbolic reasoning, provide the formulae which are needful for the mathematical description of biological phenomena for the testing of the adequacy of such description, and for the estimation of the value of such descriptions as bases of generalization from experience of a sample to the population from which the sample was drawn.

I now desire to emphasize the fruitfulness of the use of such reasoning in

the field of biology itself. Our problem is to consider whether in many instances mathematical analysis through processes of symbolic reasoning cannot supplement or in some cases largely replace other analytical methods in biology.

From the biological side the problem must be largely one of the adoption of the tested processes of the mathematician, with the substitution of constants based on the data of observation for the formal symbols of the mathematician. If it can be shown that this substitution of biological constants, and the subsequent treatment of these constants as though they were the formal symbols of the mathematician, is legitimate, the biologists power of reasoning about his observational data could be greatly extended.

Probably there are those who would categorically deny the propriety of such procedure. The final biological test of its legitimacy would seem to be two-fold: *First*, the consistency of the end results with those obtained by other methods of research; *Second*, the capacity of the methods for the prediction of the unknown.

Let me illustrate by a concrete case.

In the foregoing discussion of correlation two variables only were considered. In such cases inspection of the correlation or contingency surface may furnish graphic evidence of the existence of a relationship between the two variables. The relationship is mentally intelligible because it can be detected visually.

As a matter of fact such relationships between two sets of N paired variables represent only the simplest cases with which the biologists has to deal. Instead of considering merely x and y, he must deal with x, y and z, or with w, x, y and z, or with a far larger number of variables, all of which may be interrelated.

The complications of such a system of interrelationships are such that they cannot be readily represented graphically and they cannot be grasped by the ordinary mind. The complication is not merely that of a chess board on which moves may be made in two dimensions of space. It is the complication of a game in which the men may be moved in three or in many different dimensions of space.

Since such problems are so involved that the interrelationships cannot be visualized, they must be attacked by symbolic reasoning. Only the mathematician, of high ability and rigorous training, can write the equation which will give the probable consequences of any shift of the pieces.

For problems of this kind symbolic reasoning has led to the theory of multiple correlation and partial correlation. This in turn, has led to multiple regression or prediction equations.

For example, suppose we require to predict the basal metabolism—that is, the energy requirement per unit of time of the body when in a state of complete repose—of human individuals from various physical measurements. It is known that basal metabolism is related to sex, age, weight and stature. Individuals may be divided according to sex and equations determined for each sex separately. But even with this simplification, the problem is a relatively complex

one. We may measure in terms of the correlation coefficient the relation between age and basal metabolism, the relation between weight and basal metabolism, the relation between stature and basal metabolism. Thus all three variables, age, stature and weight must enter into our prediction equation. But these variables (age, stature and weight) are correlated among themselves, and these interrelations must be taken into account in determining equations which will give the most probable basal metabolism of an individual of known sex, age, stature and weight.

Only four variables are considered (though many others are doubtless involved) but each of these four variables may occur in many different magnitudes in the N individuals whose basal metabolism has been determined. Thus the problem becomes so complicated that no human mind can grasp or express the quantitative relations involved without the aid of symbolic reasoning.

That such multiple prediction equations will give reasonably good estimates of the values of unknown variables is proved by the results of a number of investigations in which they have been actually applied. Thus in the case of human basal metabolism it has been possible to predict, with a high degree of accuracy, the daily heat production of subjects which were unknown as far as the development of the equations was concerned.

Such examples should carry to both the biologist and the mathematician the conviction that within reason, and as established by experience, we may replace biological constants by mathematical symbols, and that these symbols, after treatment by legitimate mathematical processes, may again be replaced by their biological equivalents in our final results.

III. *Summary*

In an address which has for its purpose the outlining of the fundamental mathematical requirements of biology, it would be both undesirable and impracticable to suggest specific mathematical problems of importance for the biologist which await solution by the mathematician. I have, indeed, indicated that the greatest need is for biologists to apply more extensively the masterly series of results of Pearson and of those whom he has inspired. An enormous service mathematics has unquestionably rendered, but it has been through a few workers only. For the most part mathematicians have not felt it particularly worth their while to aid biologists in the solution of their problems.

In part, the past indifference of mathematicians has been due to the fact that biology was itself insufficiently advanced to permit the application of mathematical theory. I am inclined to think that it was in part due to the fact that the older mathematics had relatively little which could be demonstrated to be of value to the biologists and that mathematicians as a class had not yet become aware of the new fields of mathematical theory which had been developed with the specific purpose of meeting the needs of biology. Now that Quelelet, Galton and Pearson have broken down the barriers, mathematicians may find it worth their while to extend the field and to revise and refine the work already done under the stress of practical necessity.

The mathematician must not think that this problem of providing "mathematical tools" for the biologist is a simple one. In some respects it presents greater difficulties than those encountered in the field of pure mathematics.

The fact that much of the pertinent mathematical theory which is already available to biologists is still little used need not deter workers from adding to the literature. This condition of apathy on the part of biologists will not long prevail. The generation of leaders who opposed progress a few years ago will soon be without influence, and in biology we may look forward to growing appreciation and widening application of mathematical theory.

May I urge, therefore, that trained mathematicians maintain a more friendly attitude toward those biologists who are seeking through refinement of measurement and of mathematical description and analyses to place biology alongside physics and chemistry in the ranks of the exact sciences?

ON CERTAIN SEQUENCES OF CONICS AND ASSOCIATED SEQUENCES OF NUMBERS

By L. S. JOHNSTON, University of Detroit

The writer gives herein some properties of conics which he encountered while investigating the loci of the intersections of the bisectors of the angles of a triangle under certain conditions.

Theorem: *If two vertices of a triangle be taken as the foci of a conic traced by the third vertex, then the in-center and the three ex-centers trace two lines tangent to the initial conic at the vertices on the focal axis and two conics of the same species as the initial conic, concentric with the initial conic, and passing through its foci.*

Consider two points F_1 and F_2 fixed, and a variable point P tracing a conic C of eccentricity e_0, with F_1 and F_2 as foci. The bisectors of the angles at P are the tangent and the normal to C at P. If one of the foci be taken as the origin, C may be written in the form

$$\rho = k/(1 - e_0 \cos\theta)$$

or in the Cartesian forms

$$x^2(1 - e_0^2) - 2ke_0x + y^2 = k^2, \tag{1}$$

or

$$\left(x - \frac{e_0k}{1 - e_0^2}\right)^2\left(\frac{k}{1 - e_0^2}\right)^{-2} + y^2(1 - e_0^2)^{-1}\left(\frac{k}{1 - e_0^2}\right)^{-2} = 1 \qquad (e_0 < 1), \tag{2}$$

or

$$\left(x + \frac{e_0k}{e_0^2 - 1}\right)^2\left(\frac{k}{e_0^2 - 1}\right)^{-2} - y^2(e_0^2 - 1)^{-1}\left(\frac{k}{e_0^2 - 1}\right)^{-2} = 1 \qquad (e_0 > 1). \tag{2'}$$

Forms (2) and (2′) are of course identical, but are written separately to facilitate reference to the ellipse and hyperbola respectively. Throughout the paper, primed numerals will be used to designate equations which refer to the hyperbola, the corresponding unprimed numerals referring in every case to ellipses.

Using (1), the equation of the tangent to C at P is

$$x(-e_0+\cos\theta)+y\sin\theta=k$$

or

$$x(-e_0+\cos^2\tfrac{1}{2}\theta-\sin^2\tfrac{1}{2}\theta)+2y\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta=k\,; \tag{3}$$

the equation of the normal to C at P is

$$x\sin\theta-y(-e_0+\cos\theta)=\frac{e_0k\sin\theta}{(1-e_0\cos\theta)}$$

or

$$2x\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta-y(-e_0+\cos^2\tfrac{1}{2}\theta-\sin^2\tfrac{1}{2}\theta)=\frac{2e_0k\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta}{1-e_0(\cos^2\tfrac{1}{2}\theta-\sin^2\tfrac{1}{2}\theta)}\,. \tag{4}$$

The equations of the bisectors of θ and $\pi-\theta$ are respectively

$$x=\rho\cos\tfrac{1}{2}\theta,\qquad y=\rho\sin\tfrac{1}{2}\theta,\qquad \rho=+(x^2+y^2)^{1/2}\,. \tag{5}$$

and

$$x=-\rho\sin\tfrac{1}{2}\theta,\qquad y=\rho\cos\tfrac{1}{2}\theta,\qquad \rho=+(x^2+y^2)^{1/2}. \tag{6}$$

Eliminating θ between (3) and (5) we have

$$x\left(-e_0+\frac{x^2-y^2}{x^2+y^2}\right)+\frac{2xy^2}{x^2+y^2}=k,$$

whence $x=k/(1-e_0)$. This is the tangent to C at one of the vertices on the focal axis, that one for which $\theta=0$. Eliminating θ between (3) and (6) we have

$$x\left(-e_0+\frac{y^2-x^2}{y^2+x^2}\right)-\frac{2xy^2}{y^2+x^2}=k,$$

whence $x=-k/(1+e_0)$. This is the tangent to C at the other vertex on the focal axis, that one for which $\theta=\pi$. Hence those centers (in- or ex-) which lie on the tangent to C at P lie also on the tangent to C at the vertices on the focal axis. This property of conics is commonly mentioned in French books, but seems not to be often mentioned in American and English books. The analogous property of the parabola is of course familiar to the student of elementary analytic geometry.

Eliminating θ between (4) and (5) we have

$$\frac{2xy^2}{x^2+y^2}-y\left(-e_0+\frac{x^2-y^2}{x^2+y^2}\right)-\frac{2e_0kxy}{x^2+y^2-e_0(x^2-y^2)}=0.$$

Now $y=0$ only for $\theta=0$ and $\theta=\pi$, and is therefore a degenerate case; we have therefore, after reducing,

$$(7) \qquad \left(x - \frac{e_0 k}{1-e_0^2}\right)^2 \left(\frac{e_0 k}{1-e_0^2}\right)^{-2} + y^2 \left(\frac{1-e_0}{1+e_0}\right)^{-1} \left(\frac{e_0 k}{1-e_0^2}\right)^{-2} = 1 \qquad (e_0 < 1)$$

and

$$(7') \qquad \left(x + \frac{e_0 k}{e_0^2 - 1}\right)^2 \left(\frac{e_0 k}{e_0^2 - 1}\right)^{-2} - y^2 \left(\frac{e_0 - 1}{e_0 + 1}\right)^{-1} \left(\frac{e_0 k}{e_0^2 - 1}\right)^{-2} = 1 \qquad (e_0 > 1),$$

the two forms being identical, but written separately to correspond to (2) and (2′) respectively. Eliminating θ between (4) and (6) and discarding $y=0$ for reasons already given, we have, after some reduction,

$$(8) \qquad \left(x - \frac{e_0 k}{1 - e_0^2}\right)^2 \left(\frac{e_0 k}{1 - e_0^2}\right)^{-2} + y^2 \left(\frac{1 + e_0}{1 - e_0}\right)^{-1} \left(\frac{e_0 k}{1 - e_0^2}\right)^{-2} = 1, \qquad (e_0 < 1)$$

and

$$(8') \qquad \left(x + \frac{e_0 k}{e_0^2 - 1}\right)^2 \left(\frac{e_0 k}{e_0^2 - 1}\right)^{-2} - y^2 \left(\frac{e_0 + 1}{e_0 - 1}\right)^{-1} \left(\frac{e_0 k}{e_0^2 - 1}\right)^{-2} = 1 \qquad (e_0 > 1),$$

where (8) and (8′) are to be considered with (2) and (2′) respectively.

Now (7) and (8) are conics of the same species as (2), are concentric with (2), and pass through the foci of (2). The same statements hold for (7′) and (8′) in comparison with (2′). This completes the proof of the theorem.

It may be noted that the major axis of (7) is the geometric mean between the minor axis of (7) and the major axis of (8). The statement holds for (7′) and (8′) if we designate axes by the terms transverse and conjugate rather than major and minor.

The first part of the theorem just proved gives an interesting construction, viz., given the foci, the vertices, and a focal ray of a conic, to find the point at which the focal ray cuts the conic. Let V_1 and V_2 be the vertices, F_1 and F_2 the corresponding foci, and let r be a focal ray through F_2. Draw p_1 and p_2 through V_1 and V_2 respectively and normal to the focal axis; and draw through F_2 the lines b and b' bisecting the angles made by r and the focal axis. Let b intersect p_1 and p_2 at T_1 and T_2 respectively, and let b' intersect p_1 and p_2 at T_1' and T_2' respectively. Then P, the intersection of T_1T_2' and r, is on the conic and T_1T_2' is tangent to the conic at P. Similarly $T_1'T_2$ is tangent to the conic at P', the intersection of r and $T_1'T_2$. This construction exhibits the conic as the envelope of its tangents.

Now (7) and (7′) were derived by eliminating θ between (4) and (5). Hence (7) and (7′) are the loci, for the ellipse and the hyperbola respectively, of the intersections of the bisectors of θ with the normal to C at P. Similarly (8) and (8′) are the loci of the intersections of the bisectors of the supplement of θ with the normal to C at P. The vertical axis of (7) is seen to be less than that of (8), while their horizontal axes are identical. Hence (7) lies entirely inside (8) except at their points of tangency, the foci of (2). It is also easily shown that (7′) lies entirely inside (8′) except at their points of tangency, the foci of (2′).

We shall therefore call (7) and (7′) the interior derived conics of (2) and (2′) respectively, and shall designate them by IC. (8) and (8′) will be called the exterior derived conics of (2) and (2′) respectively, and will be designated by EC. I and E may therefore be considered as operators, and by their use we set up the sequences of conics and associated sequences of numbers to which the title of the paper refers. For example, if we perform on IC precisely the same operation as that by which we derived IC from C, we get the second interior derived conic of C; we shall designate this second derived conic by I^2C, defined by the identity, $I^2C \equiv I(IC)$. Repetition of this operation sets up the sequence

$$\{I^nC\} \equiv I^0C,\ IC,\ I^2C,\ I^3C,\ \cdots,\ \text{where } I^0C \equiv IC\,.$$

Similarly we make the following definitions:

$IEC \equiv I(EC)$, $EIC \equiv E(IC)$, $E^2C \equiv E(EC)$, $EI^rC \equiv E(I^rC)$, etc. It will be shown that I and E are not commutative operations, that is, that $EIC \neq IEC$.

Associated with $\{I^nC\}$ and $\{EI^nC\}$ is the sequence $\{e_n\} \equiv e_0,\ e_1,\ e_2,\ \cdots$ of eccentricities. If $e_0 > 1$, we have also, associated with the sequence $\{E^nC\}$ the sequence $\{\bar{e}_n\} \equiv \bar{e}_0,\ \bar{e}_1,\ \bar{e}_2,\ \cdots$, of eccentricities, with $\bar{e}_0 = \bar{e}_0$, but $\bar{e}_i \neq e_i$ for $i \neq 0$. This sequence will be discussed separately.

The eccentricity e_1 of (7) is given by the relation $e_1^2 = 2e_0/(e_0+1)$. This also gives the eccentricity of (7′) and (8), where e_0 is, as always, the eccentricity of the initial conic. The eccentricity $\bar{e}_1$ of (8′) is given by

$$\bar{e}_1^2 = 1 + \frac{\bar{e}_0 + 1}{\bar{e}_0 - 1} = \frac{2\bar{e}_0}{\bar{e}_0 - 1} = \frac{e_1^2}{e_1^2 - 1}\,.$$

The equation of I^rC, referred to its center as origin, is

$$a_r^{-2}x^2 + a_r^{-2}(1 - e_r^2)^{-1}y^2 = 1\,,$$

where $a_r = a_0e_0e_1e_2 \cdots e_{r-1}$; the recurrent relation of $\{e_r\}$ is

$$e_{r+1}^2 = 2e_r/(e_r + 1) \qquad (r = 0,\ 1,\ 2,\ \ldots).$$

We shall consider the two cases separately.

Case I. $e_0 > 1$.

Let ω_0 be the acute angle between the principal axis of the hyperbola and an asymptote. Then $\sec \omega_0 = e_0$ and

$$e_1^2 = 2 \sec \omega_0/(\sec \omega_0 + 1) = \sec^2 \omega_0/2\,.$$

If $\sec^2 \omega_1 = e_1^2$, we have

$$\omega_1 = \tfrac{1}{2}\omega_0 \quad \text{or} \quad \omega_r = \omega_0/2^r \qquad (r = 0,1,2,\ \ldots).$$

Then

$$\lim_{n\to\infty} e_n = \lim_{n\to\infty} \sec(\omega_0/2^n) = 1,$$

and

$$\lim_{n\to\infty} \prod_{r=0}^{n} e_r = \lim_{n\to\infty} \prod_{r=0}^{n} \sec(\omega_0/2^n) = 2\omega_0/\sin 2\omega_0 = e_0^2 \sec^{-1} e_0/(e_0^2 - 1)^{1/2}.$$

Hence the limiting form of $\{I^nC\}$ consists of the lines extending outward along the focal axis from the points whose abscissas are

$$a_0e_0^2 \sec^{-1} e_0/(e_0^2 - 1)^{1/2} \quad \text{and} \quad -a_0e_0^2 \sec^{-1} e_0/(e_0^2 - 1)^{1/2}.$$

Case II. $e_0<1$.

Let $\operatorname{sech} \omega_0 \equiv e_0$. Then $e_1^2 = 2 \operatorname{sech} \omega_0/(\operatorname{sech} \omega_0 + 1) = \operatorname{sech}^2 \frac{1}{2}\omega_0$ and, if $\operatorname{sech}^2 \omega_1 \equiv e_1^2$, $\omega_1 = \omega_0/2$, or $\omega_r = \omega_0/2^r$ $(r = (0, 1, 2, \cdots)$.
Then

$$\lim_{n\to\infty} e_n = 1, \quad \text{and} \quad \lim_{n\to\infty} \prod_{r=0}^{n} e_r = 2\omega_0/\sinh 2\omega_0 = e_0^2 \operatorname{sech}^{-1} e_0/(1 - e_0^2)^{1/2}.$$

Hence the limiting form of $\{I^nC\}$ is the segment of the focal axis between the points whose abscissas are

$$a_0e_0^2 \operatorname{sech}^{-1} e_0/(1 - e_0^2)^{1/2} \text{ and } -a_0e_0^2 \operatorname{sech}^{-1} e_0/(1 - e_0^2)^{1/2}.^2$$

The equation of EI^rC is

$$\frac{x^2}{a_{r+1}^2} + \frac{y^2}{(1 - e_{r+1}^2)^{-1}a_{r+1}^2} = 1.$$

Since $L_{n\to\infty}\, e_n = 1$, the limiting form of EI^nC is the pair of lines normal to the focal axis through the points whose abscissas are given in the preceding paragraph.

It will be noted that the vertical axes of $\{EI^nC\}$ increase without limit. The sequence of vertical axes of $\{EI^nC\}$ is not necessarily monotonically increasing, however. For, if C be an ellipse, we have the following equations:

$$EI^rC : \frac{x^2}{a_{r+1}^2} + \frac{y^2}{(1 - e_{r+1}^2)^{-1}a_{r+1}^2} = 1,$$

$$EI^{r+1}C : \frac{x^2}{a_{r+2}^2} + \frac{y^2}{(1 - e_{r+2}^2)^{-1}a_{r+2}^2} = 1.$$

It is not difficult to show that a necessary condition (also sufficient condition) for the vertical axis of the latter to exceed that of the former is that

$$7e_r^4 + 32e_r^3 + 10e_r^2 - 1 = (7e_r^2 + 4e_r + 1)(e_r + 2 + \sqrt{5})(e_r + 2 - \sqrt{5}) > 0$$

that is, that the inequality $e_r > -2+\sqrt{5}$ be satisfied. Since $L_{n\to\infty}\, e_n = 1$ for all e_0, and since $\{e_n\}$ is definitely increasing for $e_0<1$, it follows that there always

[1] See Bromwich, *Theory of Infinite Series* (1926), page 114, exercise 7. The given identity holds for imaginary as well as for real values of the argument.

[2] The reader will note that the two cases just discussed give the solution of Problem 3313 in this Monthly, vol. 35 (1928), p. 154, proposed by the writer.

exists an integer m such that $e_r > -2+\sqrt{5}$ for every $r>m$. If C be an hyperbola it is easily shown that the sequence of vertical axes of $\{EI^nC\}$ is monotonically increasing for all $e_0>1$. Therefore the vertical axes of $\{EI^nC\}$ form a monotonically increasing sequence only after that member of the sequence for which the eccentricity exceeds $-2+\sqrt{5}$.

It has been remarked that E and I are not commutative operations. If C be an ellipse, it is not difficult to set up the following equations:

$$EIC: \frac{x^2}{a_0^2e_0^2e_1^2} + \frac{y^2}{(1+e_0)(1-e_0)^{-1}a_0^2e_0^2e_1^2} = 1;$$

$$IEC: \frac{x^2}{(1+e_0)(1-e_0)^{-1}(1-e_1)(1+e_1)^{-1}a_0^2e_0^2e_1^2} + \frac{y^2}{(1+e_0)(1-e_0)^{-1}a_0^2e_0^2e_1^2} = 1.$$

Now $EIC \equiv IEC$ if and only if $(1+e_0)(1-e_0)^{-1}(1-e_1)(1+e_1)^{-1}=1$, that is, if and only if $e_0=e_1$. But $e_1>e_0$ for every $e_0<1$, as has already been shown; hence $EIC \not\equiv IEC$ if C be an ellipse. If C be an hyperbola, we have the equations:

$$EIC: \frac{x^2}{a_0^2e_0^2e_1^2} - \frac{y^2}{(e_1+1)(e_1-1)^{-1}a_0^2e_0^2e_1^2} = 1,$$

$$IEC: \frac{x^2}{a_0^2e_0^2\bar{e}_1^2} - \frac{y^2}{(\bar{e}_1+1)(\bar{e}_1-1)^{-1}a_0^2e_0^2e_1^2} = 1,$$

where $\bar{e}_1^2 = e_1^2(e_1^2-1)^{-1}$ as already defined. It is not difficult to show that a necessary condition for IEC and EIC to be identical is that $e_1^2=2$. But $e_1^2 = 2e_0(e_0+1)^{-1}<2$; hence $IEC \not\equiv EIC$, and I and E are therefore not commutative.

The sequence $\{E^nC\}$ seems less interesting geometrically than $\{I^nC\}$. If C be an hyperbola, however, the associated sequence $\{\bar{e}_n\}$ of eccentricities possesses some interest. The recurrent relation is

$$\bar{e}_{r+1}^2 = 2\bar{e}_r(\bar{e}_r-1)^{-1} \quad (r=0,1,2,\ldots), \quad (\bar{e}_0 = e_0\,;\ \bar{e}_i \neq e_i \text{ for } i \neq 0).$$

As before, let $\sec \bar{\omega}_r = \bar{e}_r$, $\bar{\omega}_0=\omega_0$ $(r=0,\ 1,\ 2,\cdots,)$. Then $\bar{\omega}_{r+1}=\frac{1}{2}\pi-\frac{1}{2}\bar{\omega}_r$. It is easy to show that $L_{n\to\infty}\,\bar{\omega}_n=\pi/3$, and also, for $\bar{\omega}_0<\pi/3$ that

$$\bar{\omega}_{2p} < \bar{\omega}_{2(p+1)} < \pi/3 \qquad (p=0,1,2,\ldots)$$

and

$$\bar{\omega}_{2p+1} > \omega_{2p+3} > \pi/3 \qquad (p=0,1,2,\ldots).$$

Hence the asymptotes of $\{E^{\bar{n}}C\}$ are alternately above and below the lines $y^2-3x^2=0$, with these lines as limiting positions.

A well known property of hyperbolas is that the sum of the squares of the reciprocals of the eccentricities of two conjugate hyperbolas is unity. From this property it follows that (8′) is similar to the conjugate of (7′). More generally stated, if C be an hyperbola, $EI^{r-1}C$ is similar to the conjugate of I^rC. In fact, if the conjugate of I^rC be reflected in the line $y-x=0$ and then be magnified by the factor $(e_r^2-1)^{-1/2}$, we shall have precisely $EI^{r-1}C$. If C be an

ellipse, $EI^{r-1}C$ can be constructed by reflecting I^rC in the line $y-x=0$ and then magnifying by the factor $(1-e_r^2)^{-1/2}$.

The theory developed for the central conics holds for the parabola if we make the customary modifications due to the fact that one focus of the parabola is at infinity. One property common to the central conics and the parabola has already been mentioned; other modifications can easily be made. It is interesting to note that the expression for the abscissa of the limiting position of the vertices of $\{I^nC\}$ is exceedingly simple in this case. If the equation of C be given in the form $y^2=4px$, it is easy to show that the equation of IC is $y^2=p(x-p)$, which is a parabola through the focus of C with focal length one fourth that of the initial conic. Hence the limit point of the vertices of $\{I^nC\}$ is the point whose abscissa is

$$p(1 + 1/4 + 1/4^2 + \ldots) = 4p/3.$$

A NOTE ON NEWTON'S DIAGRAM FOR APPROXIMATING PLANE ALGEBRAIC CURVES AND SURFACES

By ALAN D. CAMPBELL, Syracuse University

The following approach to the subject of Newton's Diagram for approximations to plane algebraic curves and to algebraic surfaces (at the origin and at infinity) has been found useful by the author. First let us consider plane algebraic curves. If an n-ic passes through the origin we solve its equation simultaneously with an equation of the form $y=x^\alpha$, then we determine α (say $\alpha=\alpha'$) so that the resulting equation in x shall have as high a power of x as possible (say x^m) coming out as a factor. Then we see geometrically that the curve $y=x^\alpha$ is a good approximation to the n-ic near the origin, since the two curves have in common there m' coincident points (where $m'=m$ if m is an integer, otherwise m' is the greatest integer less than m). If there are two or more terms in the n-ic that give us terms in x^m when the n-ic is solved simultaneously with $y=x^{\alpha'}$, and we equate to zero these terms in the n-ic, we shall have perhaps an even better approximation to the n-ic than is the curve $y=x^{\alpha'}$.

Let us consider the quintic

$$x^2y^3 - 2xy^4 + y^5 + x^2y = x^4 + y^4. \tag{1}$$

Substituting $y=x^\alpha$ in (1) we get

$$x^{3\alpha+2} - 2x^{4\alpha+1} + x^{5\alpha} + x^{\alpha+2} = x^4 + x^{4\alpha}.$$

Evidently we cannot determine α so as to have a common factor x^m in (1) with $m>4$. The origin is a triplet point on (1), with two coincident tangents given by $x^2=0$ and a third tangent $y=0$. Hence we should have two approximations to (1) at the origin. We try successively $\alpha+2=4$, $5\alpha=4$, $4\alpha+1=4$, $3\alpha+2=4$, $4\alpha=4$. The first and the fourth of these choices of α give us the approximations

we desire, since corresponding to them we have pairs of terms from the n-ic for our approximations. These two approximations are respectively $x^2y=x^4$ and $x^2y=y^4$.

To consider approximations at infinity we imitate the above procedure at the origin. For example (1) in the homogeneous form is

$$x^2y^3 - 2xy^4 + y^5 + x^2yz^2 = x^4z + y^4z. \tag{1'}$$

We see that (1′) passes through the points (1, 0, 0) and (1, 1, 0). To study (1′) at (1, 0, 0) we divide the equation through by x^5, then we put (1′) in the non-homogeneous form

$$y^3 - 2y^4 + y^5 + yz^2 = z + y^4z, \tag{1''}$$

where $y=y/x$ and $z=z/x$. We note that (1, 0, 0) is not a multiple point on the curve. Now $z=y^\alpha$ when solved with (1″) gives

$$y^3 - 2y^4 + y^5 + y^{2\alpha+1} = y^\alpha + y^{\alpha+4}.$$

We see that $\alpha=3$ gives us the best approximation. From (1″) we then obtain the approximation $y^3=z$, to which there corresponds from (1′) $y^3x^2=x^4z$. To study the curve at the point (1, 1, 0) we can change this point to (1, 0, 0) by a transformation of coordinates. If the curve (1′) passed thru (0, 1, 0) the equation we should use to replace (1″) would contain the variables x and z but no y.

Now we continue somewhat as in Hilton *Plane Algebraic Curves* §3, p. 38. Let us plot the exponents β and γ of each term $x^\beta y^\gamma$ in an n-ic as coordinates referred to a set of u- and v-axes (rectangular or oblique) obtaining what is called Newton's diagram. We suppose the u-axis to be horizontal. The equation $y=x^\alpha$ has the corresponding points (0, 1) and (α, 0). A line l thru (β, γ) parallel to the line l' joining (0, 1) and (α, 0) has the equation $(u-\beta)+\alpha(v-\gamma)=0$ and cuts the u-axis at ($\beta+\gamma\alpha$, 0). Suppose the term $x^{\beta'}y^{\gamma'}$ corresponds to a point (β', γ') on l, then $\beta'+\gamma'\alpha=\beta+\gamma\alpha$. But putting $y=x^\alpha$ in $x^\beta y^\gamma$ gives us $x^{\beta+\gamma\alpha}$ and in $x^{\beta'}y^{\gamma'}$ gives $x^{\beta'+\gamma'\alpha}$. Hence any term $x^{\beta'}y^{\gamma'}$ in the n-ic that corresponds to a point (β', γ') on l yields the same power of x as $x^\beta y^\gamma$ if we put $y=x^\alpha$, namely the power given by the u-intercept on l. Also any term $x^{\beta''}y^{\gamma''}$ where (β'', γ'') is to the left of and below l yields a power of x smaller than $\beta+\gamma\alpha$ if we put $y=x^\alpha$; whereas if (β''', γ''') is to the right of and above l the power of x yielded by $x^{\beta'''}y^{\gamma'''}$ is greater than $\beta+\gamma\alpha$. From this argument we see that if any line l joining two or more points of a Newton's diagram of an n-ic is such that all the other points of the diagram are to the right of and above l; then the terms of the n-ic corresponding to points on l, when equated to zero, give an approximation to the n-ic at the origin. Thus for (1) we have a diagram with the points (2, 3), (1, 4), (0, 5), (2, 1), (4, 0), and (0, 4). The two lines joining (0, 4) to (2, 1) and (2, 1) to (4, 0) give us the approximations at the origin that we obtained previously. Similarly we can draw a Newton's diagram for approximations at infinity. For example for (1″) we have a diagram (taking u for the exponents of the powers of y and v for powers of z) with the points

(3, 0), (4, 0), (5, 0), (1, 2), (0, 1), (4, 1). The line joining (0, 1) to (3, 0) gives us the approximation $y^3x^2=x^4z$ that we obtained above.

Next we combine all these diagrams for an n-ic (for approximations at infinity as well as at the origin) into just one diagram. We take a triangle with sides u, v, w, where w cuts u at the point $(n, 0)$ and v at $(0, n)$ and w has n equal divisions marked on it. The term $x^\beta y^\gamma$ in homogeneous coordinates is $x^\beta y^\gamma z^{n-\beta-\gamma}$. We see that $n-\beta-\gamma$ is the distance from w to (β, γ) measured parallel to u or parallel to v. To each term $x^\beta y^\gamma z^{n-\beta-\gamma}$ there now corresponds in our diagram a point whose coordinates can be written $(\beta, \gamma, n-\beta-\gamma)$ where the distance $n-\beta-\gamma$ is measured parallel to u or to v. For approximations to the n-ic at the origin we use the u- and v-axes alone, with u representing powers of x and v powers of y and with the positive directions for u and v measured from the point of intersection of u and v. For approximations at (1, 0, 0) we use as axes the u- and w-lines alone, with u representing powers of z and w powers of y, with the distance w measured parallel to the w-axis, and with the positive directions for u and w measured from the point of intersection of the u- and w-axes. Similarly for approximations at (0, 1, 0) we use w for powers of x and v for powers of z. From this discussion we have the rule that if we draw a Newton's diagram (as above) to approximate an n-ic at the origin we can use the same diagram for approximations at infinity, because any line m joining two or more points of the diagram that leaves the rest of the diagram to the left and above or below gives us an approximation at (1, 0, 0), whereas any such line n that leaves the rest of the diagram to the right and below gives us an approximation at (0, 1, 0). For example the cubic

$$x^2y + y^2z - 2xyz - xz^2 = 0 \tag{2}$$

has a diagram with the points (2, 1, 0), (0, 2, 1), (1, 1, 1) and (1, 0, 2). The line l joining (0, 2, 1) to (1, 0, 2) gives the approximation $y^2-xz^2=0$ at the origin; the line m joining (1, 0, 2) to (2, 1, 0) gives the approximation $-xz^2+x^2y=0$ at (1, 0, 0); the line n joining (0, 2, 1) to (2, 1, 0) gives the approximation $y^2z+x^2y=0$ at (0, 1, 0).

We can generalize the above discussion to the approximation of the nth degree algebraic surfaces at the origin and at infinity. (In fact the method can be generalized to any number of dimensions.) To do this we solve $x=z^\alpha$, $y=z^\beta$ simultaneously with the nth degree equation of the surface and determine α and β so as to get the highest possible power of z (say z^m) coming out as a factor. Geometrically we are finding two cylinders $x=z^\alpha$, $y=z^\beta$ such that their curve of intersection C cuts the given surface in the greatest possible number of coincident points at the origin. Now if we find all the terms $x^\lambda y^\mu z^\nu$ in the equation of the surface that yield the same power of z (namely $m=\lambda\alpha+\mu\beta+\nu$) when we put $x=z^\alpha$ and $y=z^\beta$, then the locus of these terms (when equated to zero) gives us an approximation to the surface at the origin, since this locus contains one or more curves that like C cut the given surface in the greatest possible number of coincident points at the origin.

We draw a Newton's diagram for the equation of the surface, representing each term $x^u y^v z^w$ by a point (u, v, w) referred to rectangular (or oblique) u-, v-, and w-axes. For ease of discussion we take these axes as rectangular. We take the plane ω through $(0, 0, 1)$, $(1/\alpha, 0, 0)$, $(0, 1/\beta, 0)$, namely $\alpha u+\beta v+w=1$. Through any point (λ, μ, ν) we take a plane ω' parallel to ω, namely $\alpha(u-\lambda)+\beta(v-\mu)+(w-\nu)=0$. Any point (u', v', w') on ω' gives a term $x^{u'}y^{v'}z^{w'}$ such that when we substitute $x=z^\alpha$, $y=z$ we obtain $z^{\alpha u'+\beta v'+w'}$. But from the equation of ω' we have $\alpha u'+\beta v'+w'=\alpha u+\beta v+w$, hence any points of the diagram that lie on ω' correspond to terms in the equation of the surface that give rise to the same power of z (namely $m=\alpha\lambda+\beta\mu+\nu$). The distance from the origin to the plane ω' is

$$d \equiv (\alpha\lambda + \beta\mu + \nu)/(\alpha^2 + \beta^2 + 1)^{1/2}.$$

Suppose we have a point (λ', μ', ν') to the right and above ω' or to the left and below ω'; then any plane ω'' through (λ', μ', ν') parallel to ω' has as its distance from the origin

$$d' \equiv (\alpha\lambda' + \beta\mu' + \nu')/(\alpha^2 + \beta^2 + 1)^{1/2}.$$

Since $d'>d$ or $d'<d$ respectively we have $\alpha\lambda'+\beta\mu'+\nu'$ greater than or less than $\alpha\lambda+\beta\mu+\nu$ (i.e. when we put $x=z^\alpha$ and $y=z^\beta$ in $x^{\lambda'}y^{\mu'}z^{\nu'}$ we obtain a power of z greater than or less than $m=\alpha\lambda+\beta\mu+\nu$). From the above discussion we see that to get an approximation to the given surface at the origin we must find a plane ω' through three or more points of the diagram that leaves all the rest of the diagram to the right and above ω'; then the terms in the equation of the surface that correspond to points on ω' give (when equated to zero) the desired approximation.

If the u-, v-, w-axes of a Newton's diagram are not mutually perpendicular lines we can transform them into a set of rectangular axes having the same origin and apply the above discussion. Then if we transform our coordinates back so as to refer to the original u-, v-, w-axes the above results are still valid; because by the transformations of coordinates that were employed points on a plane go into points on a plane, parallel planes go into parallel planes, and points to the right or left of any plane ω' go into points to the right or left respectively of the plane ω_1' into which ω' goes. Hence our diagram and its theory is valid also for oblique u-, v-, w-axes as well as for rectangular.

Just as for the n-ic, so for an nth degree algebraic surface we can use similar diagrams to approximate this surface at infinity at the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, and $(0, 0, 1, 0)$. Finally we wish to combine all such diagrams into a single diagram (as we did for the n-ic). We take the plane π through $(n, 0, 0)$ $(0, n, 0)$, $(0, 0, n)$ on a diagram (for approximation at the origin) that is referred to rectangular u-, v-, w-axes, namely $u+v+w=n$. Suppose π cuts the coordinate planes of the diagram in the line segments l_{uv}, l_{uw}, l_{vw}. We divide each such line segment into n equal lengths. The perpendicular distance from π to any point (λ, μ, ν) is $\rho\equiv(-\lambda-\mu-\nu+n)/\sqrt{3}$, whereas the distance from π to (λ, μ, ν) that is measured parallel to any one of the three axes is $\rho\sqrt{3}$

$\equiv n-\lambda-\mu-\nu$. Now we can use a tetrahedron with the u-, v-, w-axes and the plane π; to each point in the diagram we can assign homogeneous coordinates $(\lambda, \mu, \nu, n-\lambda-\mu-\nu)$. It is now evident that we can use π and the u-axis for a diagram to approximate the surface at (1, 0, 0, 0), π and the v-axis for (0, 1, 0, 0), π and the w-axis for (0, 0, 1, 0). In the first case we use l_{uv}, l_{uw} and the old u-axis as a set of oblique axes; in the second case l_{uv}, l_{vw} and the old v-axis; in the third case l_{uw}, l_{vw} and the old w-axis. Now we see that, having drawn a Newton's diagram with rectangular u-, v-, w-axes (oriented like the ordinary x-, y-, z-axes, respectively) to approximate an algebraic surface near the origin, we can use this diagram in the following way to approximate the surface at infinity. Any plane through three or more points of the diagram that leaves the rest of the diagram on the side toward the origin and above or below (as viewed from the u-axis by someone standing upright on the uv-plane) gives an approximation at (1, 0, 0, 0); any plane similarly placed with respect to the v-axis gives an approximation at (0, 1, 0, 0); and any plane similarly placed when viewed from the w-axis (by an observer standing on the vw-plane) gives an approximation at (0, 0, 1, 0). As an example we have the cubic surface

$$xyz + x^2y - xy^2 + x + y - 3z - 2xy + x^2z - xz^2 = 0.$$

Its diagram has the points (1, 1. 1, 0), (2, 1, 0, 0), (1, 2, 0, 0), (1, 0, 0, 2), (0, 1, 0, 2), (0, 0, 1, 2), (1, 1, 0, 1), (2, 0, 1, 0), and (1, 0, 2, 0). This gives us the respective approximations $x+y-3z=0$, $x^2y+x^2z-xz^2=0$, $y-3z-xy^2=0$, and $3z+xz^2+xy^2=0$.

LINEAR FRACTIONAL TRANSFORMATIONS ON QUARTIC EQUATIONS

By RAYMOND GARVER, University of California at Los Angeles

The purpose of this paper is to consider a number of linear fractional transformations on quartic equations from the standpoint of simple algebra. Most of the results obtained are not new, but the proofs ordinarily given make use either of invariant theory or of certain ideas of algebraic geometry, or of both.[1] We do not require a knowledge of either in this paper. Transformations on cubic equations have already been considered from the point of view of this article by Serret,[2] Niewenglowski[3] and Simonart.[4] That work, however, is much simpler, and the only interesting result seems to be that the general cubic equation can be reduced to a binomial equation by means of a linear fractional transformation. An incorrect statement, or conjecture, of Simonart to the

[1] See, for example, L. E. Dickson, *Modern Algebraic Theories*, p. 20; R. M. Winger, *Introduction to Projective Geometrie*, p. 209; E. B. Elliott, *Algebra of Quantics*, 1st ed., p. 275 ff.

[2] *Cours d'Algébre Supérieure*, 5th ed., vol. 1, p. 140.

[3] *Cours d'Algébre*, 5th ed., vol. 2, p. 309.

[4] Sur une transformation homographique d'une équation, Mathesis, vol. 36 (1922), pp. 212–218.

effect that three intermediate terms of an equation of degree greater than three can be removed by means of such a transformation led to the present paper.

We shall consider the general quartic equation in the convenient reduced form

$$x^4 + a_2x^2 + a_3x + a_4 = 0. \tag{1}$$

To this we apply the transformation

$$y = (c_1x + c_2)/(c_3x + c_4) \qquad (c_1c_4 - c_2c_3 \neq 0). \tag{2}$$

Our purpose is to so choose the parameters c_i that the transformed equation in y will be simpler than the original equation in x, in that certain of its coefficients will be zero. We thus require expressions for the coefficients of the transformed equation.

These coefficients are, of course, equal except possibly for sign to $\sigma_1 = \Sigma y_1$, $\sigma_2 = \Sigma y_1y_2$, $\sigma_3 = \Sigma y_1y_2y_3$ and $\sigma_4 = y_1y_2y_3y_4$, respectively. The second coefficient will then be zero provided we can make $\sigma_1 = 0$, the third if we can make $\sigma_2 = 0$, etc. To obtain explicit expressions for these equations of condition, we make use of the following relations:

$$y_i = (c_1x_i + c_2)/(c_3x_i + c_4), \qquad \sum x_1 = 0, \qquad \sum x_1x_2 = a_2, \qquad \sum x_1x_2x_3 = -a_3,$$
$$x_1x_2x_3x_4 = a_4.$$

The equations $\sigma_i = 0 (i = 1, 2, 3, 4)$ reduce to the following:

$$4c_1c_3^3a_4 - 3c_1c_3^2c_4a_3 - c_2c_3^3a_3 + 2c_1c_3c_4^2a_2 + 2c_2c_3^2c_4a_2 + 4c_2c_4^3 = 0, \tag{3}$$

$$6c_1^2c_3^2a_4 - 3c_1^2c_3c_4a_3 - 3c_1c_2c_3^2a_3 + c_1^2c_4^2a_2 + c_2^2c_3^2a_2 + 4c_1c_2c_3c_4a_2 + 6c_2^2c_4^2 = 0, \tag{4}$$

$$4c_1^3c_3a_4 - 3c_3c_1^2c_2a_3 - c_4c_1^3a_3 + 2c_3c_1c_2^2a_2 + 2c_4c_1^2c_2a_2 + 4c_4c_2^3 = 0, \tag{5}$$

$$c_2^4 + a_2c_2^2c_1^2 - a_3c_2c_1^3 + a_4c_1^4 = 0. \tag{6}$$

If (3) is satisfied, the second term in the transformed equation will vanish; if (4) is satisfied, the third term will vanish; if (5) is satisfied, the fourth term will vanish, and if (6) is satisfied the new constant term will vanish. We shall now attempt to choose the c_i so that two or more of these equations are satisfied simultaneously.

A natural first step is to attempt to satisfy (3) and (4) and thus arrive at the form $y^4 + b_3y + b_4 = 0$. However, this seems to require the solution of a fourth degree equation. The same reduction by means of a Tschirnhaus transformation, requiring merely the solution of a quadratic, is easily carried out. However, with respect to the quartic (or higher degree) equation, the Tschirnhaus transformation is much more general than the linear fractional.

It is possible, on the other hand, to remove the second and fourth coefficients without solving any equation of degree higher than the third. Before indicating the process it is useful to see that transformation (2) can be simplified. First,

in this and our later work we may take $c_1=1$, since it is not difficult to see that none of our results could be obtained with $c_1=0$. Second, we may also take $c_3=1$. For, in general, if the transformation $y=\phi(x)$ leads to an equation in y with certain coefficients missing, then $y=k\phi(x)$ will lead to a transformed equation with the same coefficients missing, where k is any constant. (The actual values of the non-vanishing coefficients will of course be different.) Thus if a certain group of coefficients can be removed by any transformation of type (2), they can be removed by a transformation in which $c_1=c_3=1$. Equations (3)–(6) then take the simplified form

$$4a_4 - 3c_4a_3 - c_2a_3 + 2c_4^2a_2 + 2c_2c_4a_2 + 4c_2c_4^3 = 0, \tag{3'}$$

$$6a_4 - 3c_4a_3 - 3c_2a_3 + c_4^2a_2 + c_2^2a_2 + 4c_2c_4a_2 + 6c_2^2c_4^2 = 0, \tag{4'}$$

$$4a_4 - 3c_2a_3 - c_4a_3 + 2c_2^2a_2 + 2c_2c_4a_2 + 4c_2^3c_4 = 0, \tag{5'}$$

$$c_2^4 + a_2c_2^2 - a_3c_2 + a_4 = 0. \tag{6'}$$

If we subtract (5′) from (3′), and divide out $2(c_4-c_2)$, which is not equal to zero, we obtain

$$a_2(c_2 + c_4) + 2c_2c_4(c_2 + c_4) - a_3 = 0. \tag{7}$$

And if we multiply (3′) by c_2 and (5′) by c_4, and subtract, we have

$$4a_4 - a_3(c_2 + c_4) - 4c_2^2c_4^2 = 0. \tag{8}$$

Combining these equations leads to a pair of cubic equations, one in c_2c_4, the other in (c_2+c_4):

$$8(c_2c_4)^3 + 4a_2(c_2c_4)^2 - 8a_4c_2c_4 + a_3^2 - 4a_2a_4 = 0, \tag{9}$$

$$a_3(c_2 + c_4)^3 + (a_2^2 - 4a_4)(c_2 + c_4)^2 - 2a_2a_3(c_2 + c_4) + a_3^2 = 0. \tag{10}$$

Now both of these cubics are closely related to a well-known resolvent cubic of (1):

$$r^3 - a_2r^2 - 4a_4r + 4a_2a_4 - a_3^2 = 0, \tag{11}$$

whose roots are[1]

$$r_1 = x_1x_2 + x_3x_4, \quad r_2 = x_1x_3 + x_2x_4, \quad r_3 = x_1x_4 + x_2x_3.$$

First, the roots of (9) are simply those of (11) multiplied by $-\frac{1}{2}$. And it can be easily verified, though it is less easy to derive, that the roots of (10)[2] are

$$a_3/(a_2 - r_1), \quad a_3/(a_2 - r_2), \quad a_3/(a_2 - r_3).$$

Consequently if we choose any root r of (11), c_2 and c_4 may be found by solving

[1] See, for example, L. E. Dickson, *Elementary Theory of Equations*, (1914), p. 39.

[2] I first obtained essentially this equation in a problem involving the Tschirnhaus transformation; see the Bulletin of the American Mathematical Society, vol. 34 (1928), pp. 73–74.

$$c_2c_4 = -r/2, \quad c_2 + c_4 = a_3/(a_2 - r). \tag{12}$$

This will require the solution of an additional quadratic. We thus state

THEOREM I. *The general quartic equation can be reduced to the trinomial form* $y^4+b_2y^2+b_4=0$ *by means of a linear fractional transformation, the determination of whose coefficients does not involve the solution of any equation of higher than the third degree.*

Since b_2 and b_4 could be computed explicitly without a great deal of difficulty, since the transformed equation is readily solvable and since x can be expressed in terms of y by means of (2), we have a possible method of solving quartic equations. And in some cases the solution could be performed conveniently. Thus when (11) has a rational root the process will involve only square roots. Speaking for the moment in terms of Galois groups, it is not difficult to verify that (11) will have at least one rational root if equation (1) has rational coefficients, is irreducible in the field of rational numbers, and has as its Galois group for that field one of the three possible imprimitive groups. Certain reducible quartics will also lead to an equation (11) with a rational root.

We may at once state, on the basis of our preceeding work,

THEOREM II. *The general quartic equation cannot be reduced to the binomial form* $y^4+b_4=0$ *by means of a linear fractional transformation.*

This follows at once since the left-hand side of (4′) does not become identically zero when c_2 and c_4 are chosen, as above, to satisfy (3′) and (5′). I have proved[1] the same theorem by a somewhat different method, still working, however, from equations (3), (4), and (5).

But there are certain quartics which can be reduced to binomial form; these are given by

THEOREM III. *Any quartic equation for which*[2] $T=0$ *(where* $432T$ *is defined to be* $72a_2a_4-27a_3^2-2a_2^3$*) can be reduced to the binomial form* $y^4+b_4=0$ *by means of a linear fractional transformation.*

To prove this theorem, we first note that, when $T=0$, equation (11) has the root $a_2/3$. Consequently equations (12) become

$$c_2c_4 = -a_2/6, \qquad c_2 + c_4 = 3a_3/2a_2, \tag{13}$$

and c_2 and c_4 are the roots of the equation $6a_2c^2-9a_3c-a_2^2=0$. It is now a simple matter to substitute for (c_2+c_4), $(c_4^2+c_2^2+4c_2c_4)$, and $c_2^2c_4^2$ in equation (4′) and the left-hand side actually does reduce to zero provided $T=0$. We have thus proved the theorem and at the same time showed how to set up the necessary transformation.

Finally we consider reduction to the form $y^4+b_2y^2=0$. To obtain this form

[1] Mathesis, vol. 42 (1928), pp. 163–165.

[2] T is, of course, the cubic invariant of the given quartic.

we must make three coefficients of the transformed equation zero. This is not possible in general since we have but two essential parameters, c_2 and c_4, at our disposal, with which to satisfy the three equations (3′), (5′) and (6′). We may, however, state the following:

THEOREM IV. *Any quartic equation for which* $S^3-27T^2=0$ *(where* $S=a_4+a_2^2/12$, *and* T *is defined as above*[1]*) can be reduced to the form* $y^4+b_2y^2=0$ *by means of a linear fractional transformation.*

In the proof of this theorem it is convenient to use this

Lemma. *In case* $S^3-27T^2=0$, *the two equations*

$$4c^3 + 2a_2c - a_3 = 0, \tag{14}$$

$$2a_2c^2 - 3a_3c + 4a_4 = 0 \tag{15}$$

have a common root.

The condition that these two equations have a common root is the vanishing of a certain fifth order determinant, and when the determinant is expanded the condition becomes simply that assumed in the lemma. We now choose c_2 as the common root of (14) and (15), and assume that these equations are re-written with c_2 substituted in place of c. If we now multiply (14) by c_2 and add (15), we obtain equation (6′). That is, this equation is satisfied by our choice of c_2. Similarly, if we multiply (14) by c_4 and add (15) we see that (5′) is satisfied by this same choice of c_2, without restriction on c_4. We may now choose c_4 as a root of the quadratic equation (7). From the way in which (7) was obtained we are now sure that(3′) is satisfied. Thus the proof of theorem IV is complete.

QUESTIONS AND DISCUSSIONS

EDITED BY H. E. BUCHANAN, Tulane University, New Orleans, La.

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

DISCUSSIONS

I. STANDARD DEVIATION PROBLEMS WITH RATIONAL ANSWERS

By WALTER CROSBY EELLS, Stanford University

In teaching an elementary course in statistical method it is frequently desirable, when first studying the standard deviation and the Pearson product-moment coefficient of correlation, to have available problems for which the answers are known to be rational and perhaps integral as well. This is desirable for simple illustrative problems, for class assignments, for construction

[1] S is the quadratic invariant of the given quartic, and S^3-27T^2 is its discriminant.

of tests and examinations, for preparation of alternative equivalent forms of such tests—in all of which cases it is frequently desirable to illustrate the principles with simple problems in which computation is reduced to a minimum. Many elementary students experience difficulty with square root and vary greatly in the facility with which they handle problems in which it is involved. Therefore it is useful at first to have a few problems available in which the square root can be found by inspection.

One or two of the elementary textbooks have such problems, but they are not frequent. For example Jerome[1] gives the following (with slight change of notation) as his introductory problem:

X	d	d^2	
3	3	9	
5	1	1	
6	0	0	$\sigma = \sqrt{(20/5)} = \sqrt{4} = 2$
7	1	1	
9	3	9	
Mean = 6		20	

This is composed of X's of the type

$$X = x + d \qquad (d = 1,3)$$

or more generally of the type

$$X = x + kd \qquad (d = 1,3),$$

where x = the mean, k = any rational number, integral or fractional, d = specially selected integers.

Numberless variations of this simple problem are possible. Typical ones are given in Table 1 for five selected values of x and k. For convenience a series of values like (1, 3) will be referred to as a *fundamental set.*

TABLE 1

Typical Variations of Fundamental Set (1, 3) *to Give Rational Values of the Standard Deviation.*

	$x=6$ $k=1$	$x=10$ $k=1$	$x=7.5$ $k=1$	$x=6$ $k=2$	$x=6$ $k=0.5$
$x-3$	3	7	4.5	0	4.5
$x-1$	5	9	6.5	4	5.5
x	6	10	7.5	6	6.0
$x+1$	7	11	8.5	8	6.5
$x+3$	9	13	10.5	12	7.5
	$\sigma = 2k$ $=2$	$\sigma = 2k$ $=2$	$\sigma = 2k$ $=2$	$\sigma = 2k$ $=4$	$\sigma = 2k$ $=1$

It may thus be seen that there are a doubly infinite number of variations possible for the same fundamental set, obtained by varying x or k or both.

[1] Harry Jerome, *Statistical Method*, New York, 1924, p. 156.

But are there other essentially distinct fundamental sets of the same type? It is the object of this paper to record the existence of all such sets for values of $n<10$ and of $d\leqq 10$.

Let X = original scores,

x = mean of original scores, to be selected as any desired rational value, integral or fractional,

n = number of cases, or population,

k = any rational value, integral or fractional,

d = any positive integer.

Then in the expression

$$X_i = x \pm kd_i;\ i = 1,2,3,\cdots,n;\ j = n/2 \text{ if } n \text{ is even},\ (n-1)/2 \text{ if } n \text{ is odd},$$

we seek values of the d's, such that

$$\sigma = \left(\frac{\sum k^2 d_j^2}{n}\right)^{1/2} = k\left(\frac{\sum d_j^2}{n}\right)^{1/2} = \text{a rational number}.$$

For $n<10$ and $d\leqq 10$ it can be shown that there are 385 essentially different combinations possible. By listing all of these it is found that the only sets of values of the d's having the desired property are the following:

$n=4$: One set of d's, 1, 7.
$n=5$: One set of d's, 1, 3.
$n=6$: One set of d's, 1, 5, 7.
$n=7$: Two sets of d's, 1, 2, 3; 1, 5, 10.
$n=8$: No sets of d's,
$n=9$: Three sets of d's, 1, 4, 8, 9; 1, 5, 6, 10; 2, 3, 7, 10.

The sets of measures or original scores (X's) corresponding to these d's, and the standard deviations resulting from them, are exhibited in Table 2. For convenience in this table, k has been taken as 1.

The three sets possible when $n=9$ are particularly interesting since they all lead to the same standard deviation, 6.

With the variations possible for any single fundamental set, as already illustrated in Table 1, these eight fundamental sets furnish an abundance of material for constructing problems involving the computation of the standard deviation which will "come out even." It is easy to memorize some of the combinations, especially those for $n=5$, and $n=7$. If it is desired to have the answer always integral, a sufficient condition (although not a necessary one) is that k shall be an integer. If it is desired to have the answer fractional, it is only necessary to so choose k that $2k$, $5k$, or $6k$ as the case may be shall be fractional.

In the same way in setting up simple problems for the computation of the Pearson product-moment coefficient of correlation, this table is of great value. Thus for a population of seven, one set of characteristics (x) can be assigned such numerical values that $\sigma_x = 2k$, and the other set (y) such that $\sigma_y = 6k$. Or with

a population of nine, various such sets are possible with the three fundamental sets given, although in this case both standard deviations will be 6 or some multiple of it.

TABLE 2

Sets of Individual Measures Having Rational Integral Standard Deviations ($k=1$; $n<10$; $d\leqq 10$)

$n=4$	$n=5$	$n=6$	$n=7$	$n=7$
$x-7$	$x-3$	$x-7$	$x-3$	$x-10$
$x-1$	$x-1$	$x-5$	$x-2$	$x-5$
$x+1$	x	$x-1$	$x-1$	$x-1$
$x+7$	$x+1$	$x+1$	x	x
	$x+3$	$x+5$	$x+1$	$x+1$
$\sigma=5$		$x+7$	$x+2$	$x+5$
	$\sigma=2$		$x+3$	$x+10$
		$\sigma=5$		
			$\sigma=2$	$\sigma=6$

$n=8$	$n=9$	$n=9$	$n=9$
none	$x=9$	$x-10$	$x-10$
	$x-8$	$x-6$	$x-7$
	$x-4$	$x-5$	$x-3$
	$x-1$	$x-1$	$x-2$
	x	x	x
	$x+1$	$x+1$	$x+2$
	$x+4$	$x+5$	$x+3$
	$x+8$	$x+6$	$x+7$
	$x+9$	$x+10$	$x+10$
	$\sigma=6$	$\sigma=6$	$\sigma=6$

It may be said that such problems in standard deviation or product-moment correlation are highly artificial. This is true, but the author has found them very useful in introducing these topics to classes unfamiliar with them when it is desirable to minimize the computational difficulties and to concentrate on the fundamental ideas involved. It was the feeling of need for such problems in his own classes that led to the working out of these eight fundamental sets (all of this type existing, with the limitations noted above) which furnish an abundance of such problem material. Numberless simple variations are possible, but in every case the instructor knows in advance that they will come out rational, and integral if desired, and he knows the answer without computation. Possibly other instructors in elementary statistics, in various fields, will be glad to have this source material for construction of problems in a form convenient for use.

II. Marginal Notes[1]

By T. H. Hildebrandt, University of Michigan

1. This is not a discourse on the desirability of making marginal notes in one's mathematical reading or teaching. It is simply a collection of notes of the type that every teacher who is alive to the subjects which he is teaching makes in the margins of his textbook. As a consequence there is no particular claim to originality or priority. In presenting these remarks, it is hoped that perhaps some other teacher of the first two years of collegiate mathematics may find something which will be helpful in his own teaching.

My first note refers to the area of a triangle, the coordinates of the vertices being $P_1:(x_1, y_1)$; $P_2:(x_2, y_2)$; $P_3:(x_3, y_3)$. The usual formula for this area is either a third order determinant, or the same expanded and arranged as follows:

$$\tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)],$$

I do not understand why (if third order determinants are to be avoided) this latter is preferred to the equivalent expansion

$$\tfrac{1}{2}[(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3)],$$

which has the advantage of an easy geometric interpretation and proof and which is easily remembered and easily extended. For $\frac{1}{2}(x_1y_2 - x_2y_1)$ is the directed area of the triangle OP_1P_2O, a fact which is easy to prove. Then one finds that the directed area $P_1P_2P_3P_1$ is equal to the sum of the directed areas OP_1P_2O, OP_2P_3O, OP_3P_1O. Ease in remembering is of course increased by using second order determinants, which gives in reality a method for expanding the third order determinant form.[2] By way of extension, we note that it is an easy matter to write down the area of a closed polygon. For instance the directed area of the polygon $P_1P_2P_3P_4P_5P_1$ is

$$\frac{1}{2}\left\{\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} + \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} + \begin{vmatrix} x_5 & y_5 \\ x_1 & y_1 \end{vmatrix}\right\}.$$

The geometric interpretation of the answer would in general require a figure, especially where the sides intersect between the vertices (as for instance in a five pointed star). If we write

---

[1] Read before the Michigan Section of the Mathematical Association of America, March 31, 1927, at Ann Arbor, Mich.

[2] As a side remark, I should like to voice the opinion that it would be advantageous to drop from our teaching of determinants the so-called diagonal method of expansion of third order determinants and to replace it by the expansion by minors. The diagonal method is confusing because it is not applicable to determinants of order higher than the third, as one might have a right to expect. Expansion by minors is not only applicable to determinants of any order, and hence need not be unlearned, but even for determinants of the third order is in most cases simpler than the diagonal method. Moreover, it can be made (as any one can convince himself very easily) a logical foundation for the definition of the value of a determinant of any order and can be used as a basis for proof (by induction) of the properties of determinants.

$$x_1y_2 - x_2y_1 = x_1(y_2 - y_1) - y_1(x_2 - x_1) = x_1\Delta y - y_1\Delta x$$

it is easy to see that we are not far here from the rectangular equivalent of the polar element of area, which has to a large extent disappeared from our textbooks in calculus.

The extension of these considerations to three dimensions is not so simple. Obviously a cyclic performance such as

$$\tfrac{1}{6}[\,|\,x_1y_2z_3\,| + |\,x_2y_3z_4\,| + |\,x_3y_4z_1\,| + |\,x_4y_1z_2\,|\,],$$

in which $|x_1y_2z_3|$ expresses the third order determinant on the elements x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3, is not the volume of the tetrahedron $P_1P_2P_3P_4$, though $|x_1y_2z_3|$ is the volume of $OP_1P_2P_3$. The difficulty lies in the fact that a directed volume is slightly more complicated than a directed area. It can be shown that the result desired is:

$$\tfrac{1}{6}[\,|\,x_1y_2z_3\,| + |\,x_1y_3z_4\,| + |\,x_1y_4z_2\,| + |\,x_4y_3z_2\,|\,],$$

which shows a certain amount of symmetry especially if placed in juxtaposition with the fundamental equality which we used in connection with the area of a triangle, viz.,

$$OP_1P_2 + OP_2P_3 + OP_3P_1 + P_3P_2P_1 = 0.$$

The addition of a fifth point P_5 to the melee brings up the question what volume is defined by the points $P_1P_2P_3P_4P_5$. Obviously higher dimensions increase the difficulties. We are touching here upon the notion of directed volumes in higher dimensions.

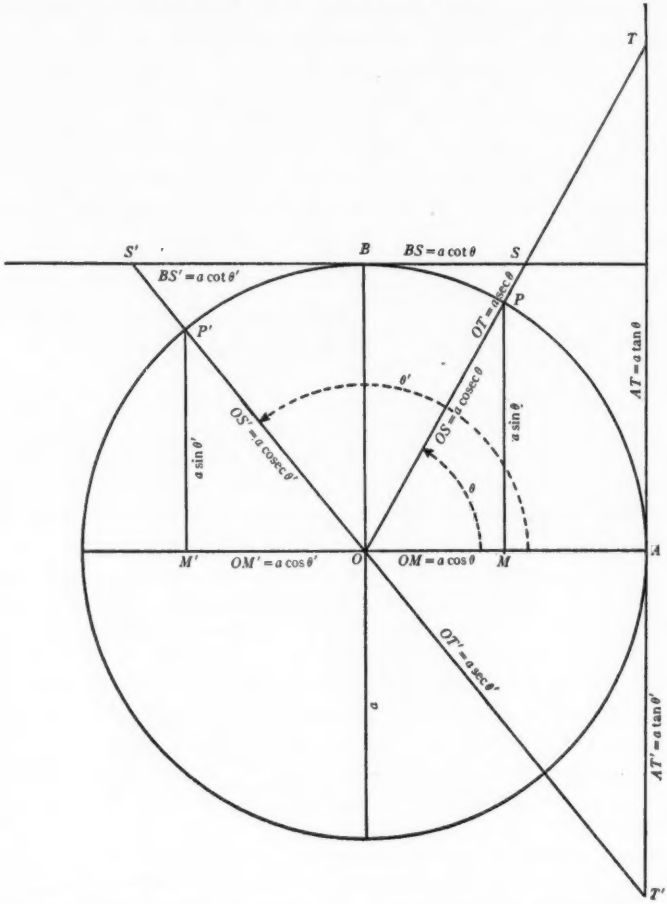
2. In the margin of the section on curve tracing in polar coordinates I find the adjoining well known (?) figure from trigonometry. It is being omitted from trigonometries to some extent, mainly because it is seldom put to use later on. If used in connection with polar coordinate paper and a pair of dividers, it yields a simple and elegant method for plotting without laborious computation enough points of curves with polar equations of the following types: $r=a\cos\theta+b$, $r=a\sec\theta+b$, $r=a\tan\theta$, $r=a\tan\theta+b\sec\theta$, to enable even freshmen to produce fine sketches. Methods may be developed for using this figure even for such curves as $r=a\cos 2\theta$, $r\cos 2\theta=a$, and others containing trigonometric functions.[1]

3. My next note occurs in connection with the remark that the tangent to a circle at a given point (x_1, y_1) on the circle can be obtained by writing down the equation of the point circle, center at (x_1, y_1) and finding the common chord of this circle and the given circle. The question arises whether this idea is extensible to conic sections and is to be answered in the affirmative. Thus the tangent to $a^{-2}x^2+b^{-2}y^2=1$ at (x_1, y_1) is the common chord of this conic and the degenerate conic $a^{-2}(x-x_1)^2+b^{-2}(y-y_1)^2=0$; the tangent to $a^{-2}x^2-b^{-2}y^2=1$

[1] More extensive remarks on the same subject are contained in a note by H. A. Bender in this Monthly, vol. 34 (1927), pp. 481–484.

is the common chord with $a^{-2}(x-x_1)^2-b^{-2}(y-y_1)^2=0$; and the tangent to $y^2=4ax$ is the common chord with $(y-y_1)^2=0$. This leads to the general statement that the tangent to

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$



at (x_1, y_1) on the curve is the common chord of this conic and the similar degenerate conic:

$$A(x - x_1)^2 + B(x - x_1)(y - y_1) + C(y - y_1)^2 = 0.$$

In addition to giving us the equation of a tangent to a conic without the necessity of finding the slope by a limits process, i.e., the use of derivatives, it has the additional advantage of making it possible to write down the equation

of the tangent plane at a given point to a quadric in three-space. For example the equation of the tangent plane to the quadric $Ax^2+By^2+Cz^2+D=0$ at the point (x_1, y_1, z_1) on the quadric is the common plane of this quadric and the similar degenerate quadric: $A(x-x_1)^2+B(y-y_1)^2+C(z-z_1)^2=0$. I leave unanswered the question of what this process gives when the point is not on the conic or quadric.

4. At a point in the proof of the fact that the locus of points, the sum of whose distances from two fixed points is a constant, is an ellipse, I find the word "stop" in the margin. If for simplicity the fixed points are $F_1:(b, 0)$, $F_2:(-b, 0)$, and the constant is $2a$, then the first equation reads:

$$[(x-b)^2+y^2]^{1/2}+[(x+b)^2+y^2]^{1/2}=2a.$$

Rationalizing[1] this equation and collecting terms ready for the second rationalization, one gets the expression to which the remark applies

$$a[(x-b)^2+y^2]^{1/2}=bx-a^2.$$

Interpreting this equation we find that it says that the distance of (x, y) from $(b, 0)$ is b/a times its distance from the line $x=a^2/b$; i.e., we have arrived at the focus-directrix property of the ellipse, which is usually the basis of discussion in analytic geometry. Obviously this observation will give us the eccentricity and equations of directrices in any case where the foci and major axis are given.

5. In connection with the section on transformation of coordinates, I find the query, "why not use on curves other than just conic sections?" For instance, in calculus, one finds occasionally such equations as $x^3-3axy+y^3=0$, and $x^2y+xy^2=a^3$, having $x=y$ as an obvious line of symmetry. If this is made the new x-axis, the equations are of relatively simple type, solvable for y^2, and in the first case, it makes it even possible to find the area of the loop of the curve without resorting to the trick of polar coordinate representation.

6. In my calculus, I find such notes as the following:

(a) $1-\cos x$ can be rationalized by multiplying by $1+\cos x$, a remark which helps to avoid the introduction of half angles in proving that $\lim_{x\to 0}(1-\cos x)/x=0$, and in the integration of expressions such as $\int(1-\cos x)^{1/2}dx\cdot\int dx/(1-\cos x)$.

(b) For some curves it is simpler to apply the fundamental definition of curvature: $d\alpha/ds$, instead of the expression in terms of y' and y''. Examples $y=\log\cos x$; the circle $x=a\cos t$, $y=a\sin t$; the four cusped hypocycloid $x=a\cos^3 t$, $y=a\sin^3 t$; the cycloid; and the involute of the circle.

(c) The tangential and normal components of acceleration of a body moving in a plane curve are the components of velocity of a point on the corresponding

[1] Professor Shohat remarked in discussing this paper that this process could be simplified by noting that $(PF_1)^2-(PF_2)^2$ is a rational expression, so that by division by PF_1+PF_2 we get PF_1-PF_2 from which PF_1 and PF_2 can be determined at once, giving the same result as above. My point is that even the straightforward usual methods have interesting interpretations.

hodograph, along and perpendicular to the radius vector to the origin. Since this vector on the hodograph is v, the polar angle α (the angle between the tangent and x-axis on original curve), the tangential acceleration is dv/dt, and the normal acceleration is $vd\alpha/dt = v(d\alpha/ds)(ds/dt) = v^2/\rho$, where ρ is the radius of curvature of the original curve.

(d) A rational function which is a proper fraction can be uniquely expressed as the sum of proper fractions having the distinct factors of the original fraction as denominators. For example

$$\frac{4x^2 + x + 5}{(x-2)^3(x^2+1)^2} = \frac{Ax^2 + Bx + C}{(x-2)^3} + \frac{Dx^3 + Ex^2 + Fx + G}{(x^2+1)^2},$$

and the coefficients in the numerator are uniquely determinable. (They are no longer unique if the fractions on the right are not proper). The emphasis being upon the *proper* fraction, it follows that the numerator is always assumed to be one degree less than that of the denominator, and we have a method covering all possible cases. Incidentally this idea does not really increase the difficulty of integration. A term like the first, with a power of a linear expression in the denominator always yields to the substitution of a new variable for the linear expression. For terms of the second type, note that for instance $x^2/(x^2+1)^2$ is easier to integrate than $1/(x^2+1)^2$. Integration by parts, or substitution of $x = a + b \tan\theta$, for a term whose denominator is of the form $[(x-a)^2+b^2]$ will always work.

(e) An expression of the form

$$e^{\alpha x}[(A_n x^n + \cdots + A_1 x + A_0)\cos\beta x + (B_n x^n + \cdots B_1 x + B_0)\sin\beta x]$$

is a solution of a linear homogeneous differential equation with constant coefficients in which the associated algebraic equation (obtained by putting $y = e^{mx}$ and dividing by e^{mx}) has the roots $\alpha \pm i\beta$ repeated $n+1$ times. This remark enables us to write down the differential equations whose general solutions are similar to

$$y = (c_1 + c_2 x) + c_3 e^{-x} + e^{3x}(c_4 \cos 2x + c_5 \sin 2x).$$

It also suggests a method for writing down the form of the particular integral for a linear differential equation with constant coefficients, in which the terms which are functions of the independent variable only, are linear combinations of expressions like the above. A particular example will illustrate best. Consider the differential equation

$$y'' - 4y' - 5y = 3xe^{-x} + 6\cos 2x + 5\sin 2x + 4e^{x}\sin 3x.$$

Since the general solution of this differential equation will also be a solution of a linear *homogeneous* differential equation with constant coefficients but of higher order (at least eighth) the solution will be determined by an algebraic equation whose roots are 5 and -1 (from the left hand side of the equation), and -1, -1, $\pm 2i$, $1 \pm 3i$ (from the right). Note that $3xe^{-x} = (3x+0)e^{-x}$ and

$4e^x \sin 3x = e^x\ (0 \cos 3x + 4 \sin 3x)$. As a consequence the complementary function is $y = c_1e^{5x} + c_2e^{-x}$ and the particular integral has the form

$$y = (c_3x + c_4x^2)e^{-x} + c_5 \cos 2x + c_6 \sin 2x + e^x(c_7 \cos 3x + c_8 \sin 3x),$$

the constants $c_3, \cdots, c_8$ being determined by the usual method of undetermined coefficients. Note that -1 counts as a triple root, as it actually is.

III. The Cheney Check Formulas

By William R. Ransom, Tufts College

After a quarter century of teaching college freshmen to solve and check right triangles, I was recently greatly pleased with the discovery by my colleague, Dr. William Fitch Cheney, Jr., of a much better check formula than any I had met during that time. From the identity

$$(c + a)[(c + b)/c] = (c + b)[(c + a)/c]$$

he obtains

$$(c + a)(1 + \cos A) = (c + b)(1 + \cos B)$$

which may be readily adapted to logarithmic computation by introducing half-angles. The logarithmic form is

$$\log (c + a) + 2 \log \cos (A/2) = \log (c + b) + 2 \log \cos (B/2).$$

Each of these formulas contains all five parts of the triangle; they contain no minus signs, are easily memorized, offer no troublesome peculiarity when the triangle is nearly isosceles or very slim, and they are equally available for all modes of solving since they use factors and logarithms not used in solving.

A Letter to the Editor

I have received a letter from Professor Julio Rey Pastor, formerly professor of mathematics in the University of Madrid and now professor at the University of Buenos Aires, in which he calls my attention to a paper "Caracteres de las formas cuadraticas definidas, con aplicacion a varias cuestiones" published by him in the Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales de Madrid, January, 1911. A reprint of the article accompanied his letter.

The paper contains a proof of the theorem on quadratic forms that appeared in my recent article in the Monthly,[1] using also the method of mathematical induction. Professor Pastor gives several interesting applications of the theorem. It seems unfortunate that the publications of mathematicians in Spain and South America are so nearly unavailable in this country.

Leonard M. Blumenthal

[1] Vol. 35 (1928), pp. 551–554.

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Hunter College, New York, N. Y.

All books for review should be sent directly to the editor of this department and not to any of the other editors or officers of the Association.

REVIEWS

Readers who are interested in the reviewing of books are invited to write to the editor of this department indicating particular books which they would like to review or the kinds of books in which they would be interested.

Introduction to the History of Science. Volume 1, from Homer to Omar Khayyam. By G. Sarton. Published for the Carnegie Institution of Washington by The Williams and Wilkins Co., Baltimore, 1927. Royal 8 vo. 12+839 pages. Price $10.00.

America's intellectual life has been enriched not a little as a result of the Great War. A notable instance is that of the author of this first volume of a work, monumental in conception, a Belgian scholar who arrived in the United States in 1915 and later adopted it as his country. For a decade he has been an Associate of the Carnegie Institution, and largely as a result of his inspiration the History of Science Society sprang into being in 1924, and the journal *Isis*, founded by Doctor Sarton in 1912, became its official organ. His great capacity for work is displayed not only by the large annual volumes of this publication and the completion of the work under review, but also by the fact that the manuscript for the second volume of the *History* is not far from being ready for the press. The period dealt with in the first volume is from the ninth century B.C. to the eleventh century A.D., and hence we do not find more than passing references to the very important ancient civilizations of Egypt, Babylon, and China.

The author looks upon the development of science as the development of systematized positive knowledge and thus one is impressed with his breadth of outlook even though the omission of discussion in some fields and the inclusion of others might at first be questioned. But because excellent works of reference are already available there are few references to such things as political history, economic history, and history of art. On the other hand there is discussion of the history of music, which for a millenium was regarded as a part of mathematics; and much space is devoted to the history of religion, since the author was not aware of any work giving in chronological order an account of the religious experiences of mankind. The early history of philology is also regarded as of considerable importance. For, as the author remarks, "the discovery of the logical structure of language was as much a scientific discovery as, for example, the discovering of the anatomical structure of the body."

In further defining the field of his inquiry Doctor Sarton contemplated at first its limitation to pure science but he met with difficulties in separating consideration of pure science from that of its applications. It was finally decided to refer, for examples, to "a physician, an engineer, or a teacher only if he added something definite to our knowledge, or if he wrote treatises which

were sufficiently original and valuable, or if he did his task in such a masterly way that he introduced new professional standards."

The work is divided into a long introductory chapter, and 34 other chapters each dealing with a particular epoch and labelled by a descriptive title including, for the most part, the name of an outstanding individual of the epoch: thus (I) "The Dawn of Greek and Hebrew Knowledge (ninth and eighth centuries B.C.)"; (V) "The Time of Plato (first half of fourth century B.C.)"; (VIII) "The Time of Archimedes (second half of third century B.C.)"; (XXII) "The Time of Proclos (second half of fifth century)"; (XXVII) "The Time of Bede (first half of the eight century)"; (XXIX) "The Time of al-Khwarizmi (first half of ninth century)"; (XXXIV) "The Time of Omar Khayyam (second half of eleventh century)". After the fourth chapter each chapter covers a half century so that chapters V to XXXIV deal with the period from the fourth century B.C. to the end of the eleventh century of our era.

To give an idea of the kind of material one may find in a chapter, some indication of the contents of chapters VIII and XXII will be given. In chapter VIII there are nine main sections: 1. Survey of Science in the second half of the third century B.C.; 2. Development of Buddhism in India; 3. Unification of China; 4. Chinese and Greek philosophy; 5. Greek mathematics, astronomy, and physics (the contributions of Archimedes, Eratosthenes, Conon, and Apollonios of Perga are considered); 6. Introduction of Greek medicine in Rome; 7. Chinese and Hellenistic technology; 8. Greek and Roman historiography; 9. Chinese writing. With each section there is a bibliography; in case of Apollonios this fills more than a page, and of Archimedes more than two pages. (Dr. Sarton attributes an edition of the works of Archimedes to Kliem who was in this connection only a translator of Heath's edition.) Throughout the different chapters the bibliographies, with many critical comments, constitute an exceedingly valuable feature of the work.

In chapter XXII, the time of Proclos, the nine subheadings are: 1. Survey of Science in the second half of the fifth century; 2. Religious background (the Talmud); 3. Hellenistic, Syriac, and Latin philosophy (Proclos, and half a dozen others); 4. Latin, Hellenistic, and Hindu mathematics (Domninos, Metrodorus, Āryabhata; Doctor Sarton will probably want to refer in a new edition to Datta's "Two Āryabhatas of Al-Biruni," *Bull. Calcutta Math. Soc.*, 17, 1926, p. 60–74; and to A. G. Laird's, "Plato's geometrical number and the comment of Proclus," Madison, Wis., 1918); 5. Latin Hellenistic, Chinese, and Hindu Astronomy; 6. Chinese Geography; 7. Latin and Singalese historiography; 8. Roman and barbarian law; 9. Chinese philosophy.

While such scholars as Steinschneider, Suter, Vincent, and Woepcke finely contributed to our knowledge of Muslim mathematics, the work which Doctor Sarton is publishing contains the first fairly complete account of mediaeval science as a whole. Why the Muslims were so far ahead of the Christians from the eighth to the eleventh centuries is set forth clearly. The Greek tradition

in connection with disinterested research was stifled by Roman utilitarianism followed by theological expediency, and "theological domination, which seemed for a long while to destroy every hope of a genuine scientific revival." On the other hand Persianized Muslims enthusiastically studied and imitated the intellectual structures reared by the Greeks, and from Spain to central Asia were developed centers radiating Muslim culture.

The student and teacher of the history of mathematics will find Doctor Sarton's very valuable work suggestive at almost every turn, and notably supplementing material available in other sources. It is heartily recommended both to the amateur and to the scholar.

R. C. ARCHIBALD

Algebra. By Oskar Perron. Walter de Gruyter and Co. Berlin and Leipzig, 1927. I: Die Grundlagen, VIII+307 pages. II: Theorie der algebraischen Gleichungen, viii+243 pages.

These two volumes, 8 and 9 of the Göschen series in pure mathematics, are from the pen of the author of the first volume. With each addition to the series these useful expositions seem to acquire a more substantial character. It is the express purpose of the author to write a book which, though designed primarily to meet the needs of advanced students, shall also be of service to the investigator. This double aim has been admirably accomplished for the book is more than a transcription, with pedagogic alterations or the more common pedagogic deletions, of existing treatises. A fair share of the important duty of putting modern mathematical research into connected exposition has been skillfully done.

In the view of the author algebra is that part of analysis which is based on the rational operations rather than on the notions of relative magnitude and of limit. The only important departures from this limitation are those inevitably involved in the proof of the fundamental theorem that every equation with coefficients in the field of complex number has a root belonging to that field, and in the numerical calculation of the roots. The notion of field is introduced at the outset and theorems are stated thereafter with reference to their field of validity.

The student of the subject will be attracted by the orderly and careful development, and by the sympathetic guidance afforded by the illuminating comment at various stages of progress. The historical references are adequate though not especially numerous.

In the first volume the existence of the real numbers as developed by the author in the initial volume of the Göschen series is assumed. After the introduction of complex number, number fields and number rings are defined. These concepts are then applied to polynomials. A second chapter on formal processes is followed by two of conventional type on determinants and symmetric functions. The last two chapters headed respectively "divisibility" and "existence of the roots" are of more novel character. The discussion of dependent polynomials

and of resultants, with finally a proof of Bézout's theorem on the number of solutions of k equations in k variables, based on relatively recent work of Mertens, is most satisfactory. Certain familiar topics such as elementary divisors, and the equivalence of pairs of bi-linear forms are not mentioned. For these topics, already well handled in current texts, the above substitutions are very welcome.

The second volume contains preliminary chapters on the numerical solution of equations and the solution of particular equations. A chapter on substitution groups is followed by an exposition of the Galois theory of equations. The volume concludes with a chapter on the equation of the fifth degree. The author first gives a method by which the general quintic equation can be transformed into the Brioschi normal form by the adjunction of square roots alone. According to a verification which he makes but does not reproduce this method fails only in trivial metacyclic cases. The sextic resolvent of Jacobi is then solved by "elliptic functions" rather than elliptic modular functions. The elliptic functions actually used are the values of the p-function for fifths of the periods, which in the terminology of Klein-Fricke are modular forms. From a function-theoretic standpoint such forms are not as simple as modular functions but, as Perron points out, they are more easy of access. Finally the roots of the Brioschi quintic are expressed in terms of the roots of the Jacobi resolvent.

ARTHUR B. COBLE

An Introduction to Linear Difference Equations. By Paul M. Batchelder. Published with the cooperation of the National Research Council. The Harvard University Press, 1927. viii+209 pages.

The development of the theory of linear difference equations for a long time lagged far behind that of the related field of linear differential equations. This fact was due to certain intrinsic difficulties in the former which called for the introduction of new ideas and methods. Some of these ideas and methods were introduced by Poincaré as far back as 1885 and 1886 (American Journal of Mathematics and Acta Mathematica) in connection with his development of the theory of asymptotic representation of functions by means of power series. But it was in the years 1909 to 1912 that the theory of difference equations began to take satisfactory form from the function-theoretic point of view. In these years comprehensive existence theorems were first developed and published almost simultaneously by workers in Denmark and America and France.

In recent years there has been a rapid development of the theory. Prior to the publication of the book under review only one treatise had appeared taking account of these recent advances, namely, *Vorlesungen über Differenzenrechnung* by N. E. Nörlund. Notwithstanding the fact that this book puts much greater emphasis upon some parts of the development than upon others of equal importance, it will probably remain for some time the most valuable treatment of the subject as a whole.

The work of Batchelder makes no attempt to be a comprehensive treatise;

in fact, its purpose is quite different. Batchelder contrasts his purpose with that of Nörlund in the following passage from the preface:

> Up to the present time no attempt has been made to provide the student with a convenient introduction to this new field, in which so many problems still await solution. The only book which deals with the theory from the modern standpoint is the recent one of Nörlund; while this is of great value to the advanced student in furnishing a general view of the literature of the subject, its failure to give a systematic presentation of the elements of the theory renders it unsuitable for the beginner. The aim of the present book is in part to fill this need by affording as simple and direct an approach as possible to the fundamental facts and ideas of the theory, and in part to extend the boundaries of the subject by studying certain important exceptional cases which have hitherto defied analysis.

Knowledge of the elements of the theory of functions of a complex variable is sufficient to enable the student to read the book of Batchelder. In the first chapter (pp. 1–31) several preliminary topics are treated and the reader becomes acquainted with some of the fundamental concepts and methods which he will need later. The second chapter (pp. 32–67) is devoted to equations of the first order. In particular, it contains a development and some applications of the theory of the gamma functions. In the first two chapters exercises are added for the use of the student. The remaining two chapters are devoted to the hypergeometric difference equation, that is, the linear homogeneous equation of the second order with linear coefficients. The third chapter (pp. 68–126) deals with the so-called general case. The irregular cases are treated in the fourth chapter (pp. 127–209). Unfortunately there is no index.

The treatment throughout, which is inspired principally by the methods of Birkhoff, is based mainly on the divergent power series which formally satisfy the difference equation; but use is also made (to a lesser extent) of certain methods of Nörlund. The first three chapters are the more valuable ones for the learner. The last chapter contains useful contributions to the theory. In the opinion of the reviewer, the book would have been more useful if the main content of the last chapter had been published in a memoir and the space thus saved had been devoted to other more general introductory aspects of the theory. But the first three chapters will probably furnish the reader with a sufficient introduction to the subject to enable him to pass with comfort to a study of the original memoirs. It is in this that the chief value of the book lies. Since it was not intended to be a comprehensive treatise on its subject, this is the most important use which it can serve; and it serves well such a purpose. By aid of this book it now becomes possible for the first time to obtain from the printed page (that is, from books and memoirs) a comfortable introduction to the theory of linear difference equations. On this account the book will be a very useful one.

This is not the place to give a detailed critical analysis of the exposition. In the main the treatment is satisfactory, though it is not always entirely free from criticism.

Notwithstanding the excellencies of this book and that of Nörlund and the fact that they supplement each other in useful ways, and notwithstanding the

great richness of content of Nörlund's book, there is still a distinct need for a comprehensive exposition of the theory of the difference calculus—an exposition which shall adequately present the principal researches already achieved and shall also fill up not a few lacunae now existing in the theory. The need of supplying these deficiencies makes it a difficult matter to prepare just the sort of book that is required; but the need remains as a challenge to the workers in this field.

R. D. CARMICHAEL

Plane and Spherical Trigonometry. By Jabir Shibli. Ginn and Company, 1928.

Professor Shibli has written a textbook on trigonometry that will make the subject a pleasure to the learner and a joy to the teacher. The matter contained in his book does not differ from that of dozens of other texts, but the manner of the presentation distinguishes it. The approach to the various topics is made in a fashion so attractive as to enlist the students' interest, and interest is the keynote of success in any study. With illustrations by simple graphic solutions of problems the learner is led on to understand the functional relations of the parts of a triangle and to proceed with confidence to the application of them in trigonometric analyses.

Some parts of his text, notably Chapter XI, "Complex Numbers," are rather sketchy; but the teacher can easily fill up the gaps if it is thought best to treat this matter in a freshman course in trigonometry. Long lists of problems of a nature to appeal to the student's interest are given at frequent intervals, from which the teacher may assign work suitable to all degrees of ability and industry of his students. The book is pedagogically sound and will no doubt meet with the favor it deserves. The publishers have given it a setting worthy of the book. It is a pleasure to see a textbook whose outward dress conforms so simply and yet so tastefully with its contents.

GEORGE A. HARTER

Mathematics in Liberal Education. By Florian Cajori. The Christopher Publishing House, Boston, 1928. 169 pages. Price $1.50.

It seemingly has been the habit of the centuries to challenge the value of mathematics in the school curriculum. Hence there have been many expressions of approval or disapproval from the greatest men of all ages. Professor Cajori, in his book, "Mathematics in Liberal Education," has collected this testimony and made it available for those who are sincerely interested in the subject. This book he has "dedicated to the advancement of sound education in America." On page 19, the author states his purpose thus: "We are endeavoring to take as complete a survey as possible of writings on the educational value of mathematics. We have taken great pains to collect judgments on both sides of the question. We have not selected our witnesses. We have given by name and have counted all writers known to us who have expressed themselves on the

mind-training value of mathematics. In no case have we suppressed the name of a writer whose testimony we found to be against mathematics."

Of course, the mathematician will be impressed by the obvious fairness of the plan and even the modern psychologist (if he believes that the witnesses were not "picked") will be forced to admit that Professor Cajori has made a very strong case for the great value of mathematics in a liberal education. The testimony of the witness is given in his own words and the comments of the author are brief and impartial, and cause one to feel that each witness is being given a fair chance to speak for himself.

The witnesses are divided into six groups: Greek, Roman and middle ages, fifteenth and sixteenth centuries, seventeenth and eighteenth centuries, nineteenth century, twentieth century. After each group is given a brief summary, the character of which may be illustrated by this quotation from the first one: "We find, therefore, that Protagoras opposed mathematics as necessary for mind training, while Plato, Aristotle, . . . , Plutarch, and Proclus stressed its great educational value. The count is 12 to 1."

The closing chapter gives a table, showing the statistical results of the preceding chapters, together with an excellent study of these results. Of those quoted, 603 favored the study of mathematics and 128 opposed it. Even if the testimony of the 195 teachers of mathematics be excluded because of bias, the vote would still be 413 to 123. It is interesting to note that though the average vote is about 5 to 1 in favor, the vote was closest in the seventeenth and eighteenth centuries when it was 103 to 30.

The book is attractive in appearance and the reader will find it most convenient for reference, as it is arranged chronologically and has a good index. Professor Cajori has done a fine piece of work that the young teacher will find particularly helpful at this time.

R. P. STEPHENS

Trigonometry. By A. W. Siddons and R. T. Hughes. Cambridge University Press, 1928. viii+320 pages.

If an instructor famed as a driver of students were allowed by his department and dean to give a hundred and twenty assignments from the English text under review to a selected class of American collegians, then that class would learn trigonometry as do few classes in this country. American colleges do not pretend to teach the amount of trigonometry contained in this book, written by two masters at Harrow and presumably designed primarily for use at the English public schools. By omitting four examples out of five, a class meeting fifty times could cover fairly adequately the first two parts of the text; the third part, dealing for some sixty pages with complex numbers, exponential and hyperbolic functions, and series, presupposes some mathematical maturity, and in several places a knowledge of the differential calculus.

Part I is concerned with the solution of triangles, and is hence called "numerical" trigonometry. The six trigonometric ratios are defined first for

acute angles, and later for obtuse angles. The pupil is supposed to be already familiar with logarithms. Tables, by the way, are not included in the text. "Algebraic" trigonometry forms the second part, running a hundred and sixty pages. The usual topics of general angle, compound angle, radians, identities and equations, periodic functions etc., are well handled; in addition there is some geometry of the triangle and quadrilateral beyond the powers of most of our freshmen. This second part of the text is thought by the authors to be indispensable to all students of science and engineering. Part III, whose contents we have mentioned, was intended for "serious" students of science and engineering.

The English believe in teaching elementary mathematics by problems. Think of 1870 problems, over a third consisting of several parts, all with answers, (the answers are in some thirty pages at the end of the book), and, praise Allah!, carefully grouped under such headings as right triangles, viva voce, heights and distances, traverses and projections, three dimensions, symbolical, sine and cosine rules, harder problems in three dimensions, etc., plus a large number of fully displayed illustrative problems, especially in the first part. There are twenty-three "Revision Papers," which turn out to be very well constructed two and three hour examinations. The reviewer's task of making out a final examination in trigonometry will be materially lessened for the next twenty three years.

Turning to minor details, the typography is excellent. There appear to be no misprints. The answers given seem reliable, for some five per cent of them, taken at random, were checked by the reviewer and several of his colleagues. The King's English occasionally sounds strange to American ears. Few freshmen girls could repress a snicker when they read on p. 17 "you would think of 12/c" or on p. 141 "you may be inclined to argue." The truth is that Messrs. Siddons and Hughes often write as they teach, somewhat informally. Frequently their footnotes are addressed directly to the student, and very helpful they are.

Opinions will likely vary about the possibility of using this English text with American freshmen; one would think that an experienced teacher could readily make the necessary modifications. As a collection of problems the volume is a masterpiece.

C. A. RUPP

Questions D'Arithmétique. Par B. Niewenglowski. Vuibert, Paris, 1927. viii+225 pages.

This book consists mainly of very clearly presented solutions of elementary problems and proofs of some elementary theorems in the theory of positive integers. As examples of these we may mention the following: to resolve in integers the equation $x^2+y^2=z^2$; to find the greatest common divisor of x^m-y^m and x^n-y^n when x, y, m, n are integers; to determine all even perfect numbers; if x^2+2y is a square then x^2+y is the sum of two squares; in order that a number p shall be a prime it is necessary and sufficient that there shall be one and just

one square x^2 such that $p+x^2$ is a square; if p is a prime and x belongs to the exponent 3 modulo p, then $x+1$ belongs to the exponent 6 modulo p. Sums of powers of consecutive integers are treated and the results are applied in the resolution of elementary problems. It is to such fragments that the book is mainly devoted. But it contains also a derivation of the law of quadratic reciprocity and a brief and incomplete account of the so-called Pell equation. The book makes no contribution to the theory of numbers; it seems to be intended only for amateurs. But it will afford a source for very easy problems suitable for use in an undergraduate course involving the most elementary parts of the theory of numbers.

R. D. CARMICHAEL

A History of Mathematical Notations. Volume I. Notations in Elementary Mathematics. By Florian Cajori, Ph.D. The Open Court Publishing Company, Chicago, 1928. xvi+451 pages. Price $6.00.

The aim of this work as stated in the Introduction is "to give not only the first appearance of a symbol and its origin (whenever possible), but also to indicate the competition encountered and the spread of the symbol among writers in different countries." The author states that he "believes that this history constitutes a mirror of past and present conditions in mathematics which can be made to bear on the notational problems now confronting mathematics."

In view of a present day movement to prepare for students of the history of the sciences, a series of "Source Books" which shall include the first appearance of epoch-making concepts, the publication of this work on mathematical notations is especially timely. It is a veritable source book for symbols in elementary mathematics. This aspect of the work may be illustrated by the three-page *Extract from Miscellanea Berolinensia*, due to Leibniz, followed by the translation. It may be illustrated also by the treatment accorded William Oughtred. This English mathematician used one hundred and fifty symbols, among them the present common sign for multiplication. Five pages are devoted to these symbols with an indication of the editions of the works in which they are found, all this followed by copious notes. *The History of Mathematical Notations* may be consulted for authors who used a symbol to represent any of the elementary notions of mathematics, for these individual writers are fully discussed; or the book may be consulted for the detailed history of any symbol.

There is a richness of explanation throughout the work which can only be touched upon. It is shown in the fascinating history of the symbol $\times$ in multiplication, a history which gives evidence of the thoroughness of the research here represented. It is shown again in the discussion of Pacioli as an individual writer. This is made from a direct study of the 1494 editions of Pacioli's great work. The author shows here a perfect familiarity with the ground that he is covering and displays his mastery of interpretation and his ability to coordinate. Such a treatment leads to an expression of the wish that the author had had

access to more original sources. A very complete study of the role of Pitiscus in the use of decimal fractions and the dot as a decimal separatrix is made from the 1612 edition of the *Trigonometria* with a presentation of the case leading to the conclusion that Fitiscus made extended use of decimal fractions but that the honor of introducing the dot as the separatrix between units and tenths must be assigned to others. Had the author been able to make the same convincing study of Edward Wright's 1616 edition of Napier's *Descriptio* and of Napier's *Rabdologiae . . . . Libri Dvo* of 1617, the reader would not experience the feeling of passing from a reliable exposition to an indifferent treatment of the places these two men take in the controversy; but it is doubtless impossible to avoid unevenness in a work of such magnitude as the one under consideration.

There is such a wealth of reliable material in this book that it is with some hesitation the matters which follow are noted. However, attention should be called to the fact that there are some misleading statements and illustrations. The statement that "these signs [*Herodianic signs*] appear on an abacus found in 1847, represented upon a Greek marble monument on the island of Salamis" does not square with the fact that the Salamis abacus is now in the Epigraphical Museum at Athens. Following this is a drawing (p. 23) with the title, "The computing table of Salamis," which departs from the original in several respects. On the latter there are no counters, as shown here, and the Greek number symbols do not appear alongside the set of lines containing the crosses but to the right of them and running over the shorter lines. The illustration (p. 79) from the Bakhshālī Manuscript taken from G. R. Kaye, *Indian Mathematics* (1915), is in an inverted position to the one given in that work. Again, since so much must be omitted from a treatise like this, the voluminous treatment of Hindu-Arabic numerals with several pages devoted to "Fanciful hypotheses on the origin of the numeral forms" might well have been shortened to make room for a more extended history of such a topic as rod numerals.

As page follows page in this work, there emerges the story of the gradual adoption of the symbols now in use. It is not an evolution but a survival for one reason or another of a particular form. The conclusions to be drawn are stated forcibly in the remarks made by the author at the close of the history of the symbol for subtraction. He says: "This study emphasizes the difficulty experienced even in ordinary arithmetic and algebra in reaching a common world-language. Centuries slip past before any marked step toward uniformity is made. . . . . The only hope for rapid approach of uniformity in mathematical symbolism lies in international cooperation through representative committees."

The contents are: "Numeral Symbols and Combinations of Symbols;" "Symbols in Arithmetic and Algebra" (Elementary Part), treated under "Groups of Symbols Used by Individual Writers" and "Topical Survey of the Use of Notations;" "Symbols in Geometry" (Elementary Part), treated under "Ordinary Elementary Geometry" and "Past Struggles between Symbolists and Rhetoricians in Elementary Geometry." The illustrations number 106 and include some entirely new ones such as the page from Klebetius and the page

from the Bolognetti manuscript, of possibly 1550, containing an equality sign almost identical with that of Robert Recorde. Innumerable footnotes throughout the work provide a bibliography of great value. There are also cross references in the text which add materially to the convenience of the reader and to the smoothness with which a study of it proceeds.

The Open Court Publishing Company has added another notable achievement to the long list of works made available through the courtesy of this unique house.

LAO G. SIMONS

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. L. OLSON

Send all communications about Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscripts should be typewritten, with double spacing and with a margin at least one inch wide on the left.

PROBLEM FOR SOLUTION

N. B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in the Monthly. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.

3372. *Proposed by Nathan Altshiller-Court, University of Oklahoma.*

Given a point of a conic, the tangent at this point, an axis, and the center (at a finite, or infinite distance), the following two lines are constructed: (1) the symmetric of the given tangent with respect to the parallel to the given axis through the given point, and (2) the diameter through the symmetric of the given point with respect to the axis. Prove that if at the point of intersection of these two lines the perpendicular is erected to the first line, it will meet the normal to the conic at the given point in the center of curvature of the conic at this point.

3373. *Proposed by Paul Capron, U. S. Naval Academy.*

A plane cuts the lateral edge of a quadrangular pyramid so that the ratios of the edges of the whole pyramid to the corresponding edges of the smaller pyramid cut off are x_1, x_2, x_3, x_4 and the ratio of the smaller volume to the larger is r. Either diagonal of the base of the larger pyramid divides this base into triangles whose areas bear the ratio r_i, r_{i+2} to the whole area of the base, each of these ratios being numbered to correspond with the lateral edge opposite its triangle, so that $r_1+r_3=1=r_2+r_4$. Show that

$$x_1x_2x_3x_4r = x_1r_1 + x_3r_3 = x_2r_2 + x_4r_4.$$

3374. *Proposed by J. Rosenbaum, Milford, Conn.*

Given two triangles, one within the other, to construct a third triangle which shall be inscribed in the outer and circumscribed about the inner triangle. Also prove that if the two given triangles are equilateral and concentric the third triangle is equilateral.

3375. *Proposed by A. S. Levens, University of Minnesota.*

Show to find the angle between a line and a plane graphically, without first reducing the problem to that of finding the angle between two lines.

SOLUTIONS

3310 [1928, 154]. *Proposed by B. F. Finkel, Drury College.*

Find the envelope of a system of circles having for diameters the secants of constant length, $2r$, of a conic.

Solution by W. J. Patterson, The University of Western Ontario

Let equation of central conic be

$$b^2x^2 + a^2y^2 - a^2b^2 = 0. \tag{1}$$

Let equation of secant to conic be

$$y = mx + k, \tag{2}$$

and let P_1, P_2, be the points of intersection with conic. Then

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = 4r^2. \tag{3}$$

Therefore, the equation of the variable circle is

$$\{x - \tfrac{1}{2}(x_1 + x_2)\}^2 + \{y - \tfrac{1}{2}(y_1 + y_2)\}^2 - r^2 = 0. \tag{4}$$

Using equation (2) in equation (1), we get

$$(x_1 + x_2)/2 = -a^2mk/(a^2m^2 + b^2),$$

$$x_1x_2 = a^2(k^2 - b^2)/(a^2m^2 + b^2). \tag{5}$$

Similarly,

$$(y_1 + y_2)/2 = b^2k/(a^2m^2 + b^2),$$

$$y_1y_2 = b^2(k^2 - a^2m^2)/(a^2m^2 + b^2). \tag{6}$$

Combining (5) and (6) in (3), and clearing of fractional forms,

$$a^4m^2k^2 - a^2(k^2 - b^2)(a^2m^2 + b^2) + b^4k^2 - b^2(k^2 - a^2m^2)(a^2m^2 + b^2) - r^2(a^2m^2 + b^2)^2 = 0. \tag{7}$$

Also, combining (5) and (6) in (4), we have, simplified,

$$(x^2 + y^2)(a^2m^2 + b^2)^2 + 2a^2mk(a^2m^2 + b^2)x - 2b^2k(a^2m^2 + b^2)y + k^2(a^4m^2 + b^4) - r^2(a^2m^2 + b^2)^2 = 0. \tag{8}$$

Eliminating k between (7) and (8), arranging as a power series in m, and deriving the discriminant relation in the usual way, we obtain the equation of the required envelope.

The resulting equations are so complicated that it does not seem worth while to write them.

Also solved by William Hoover.

3311 [1928, 154] *Proposed by A. Pelletier, Montreal, Canada.*

Eliminate $x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3$ from the following equations:

$$a^{-2}x_1^2 + b^{-2}y_1^2 + c^{-2}z_1^2 = 1, \qquad a^{-2}x_2^2 + b^{-2}y_2^2 + c^{-2}z_2^2 = 1,$$

$$a^{-4}x_1x_2 + b^{-4}y_1y_2 + c^{-4}z_1z_2 = 0, \qquad a^{-4}x_2x_3 + b^{-4}y_2y_3 + c^{-4}z_2z_3 = 0,$$

$$a^{-2}x_1x + b^{-2}y_1y + c^{-2}z_1z = 1, \qquad a^{-2}x_2x + b^{-2}y_2y + c^{-2}z_2z = 1,$$

$$a^{-2}x_3^2 + b^{-2}y_3^2 + c^{-2}z_3^2 = 1,$$

$$a^{-4}x_1x_3 + b^{-4}y_1y_3 + c^{-4}z_1z_3 = 0,$$

$$a^{-2}x_3x + b^{-2}y_3y + c^{-2}z_3z = 1.$$

Solution by Frank Ayres, Jr., College Station, Tex.

Write the equations

$$a^{-2}x_i^2 + b^{-2}y_i^2 + c^{-2}z_i^2 = 1, \tag{1}$$

$$a^{-2}xx_i + b^{-2}yy_i + c^{-2}zz_i = 1, \tag{2}$$

$$a^{-4}x_ix_j + b^{-4}y_iy_j + c^{-4}z_iz_j = 0, \tag{3}$$

where $i, j = 1, 2, 3; i \neq j$.

(1) States that the three points (x_i, y_i, z_i) are on the ellipsoid, $a^{-2}x^2 + b^{-2}y^2 + c^{-2}z^2 = 1$.

(2) gives the equations of the tangent planes to the ellipsoid at the points.

(3) states that these planes are mutually perpendicular.

If we write

$$r_i = a^{-2}x_i(a^{-4}x_i^2 + b^{-4}y_i^2 + c^{-4}z_i^2)^{-1/2} \equiv a^{-2}x_i\omega_i^{-1};$$

$$s_i = b^{-2}y_i\omega_i^{-1}; \qquad t_i = c^{-2}z_i\omega_i^{-1},$$

(2) becomes

$$r_ix + s_iy + t_iz = (a^2r_i^2 + b^2s_i^2 + c^2t_i^2)^{1/2}. \tag{4}$$

Squaring and adding, we have, since

$$\sum r_i^2 = \sum s_i^2 = \sum t_i^2 = 1; \qquad \sum r_is_i = \sum s_it_i = \sum t_ir_i = 0,$$

the required result,

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2.$$

This locus is the director sphere of the ellipsoid.

Also solved by F. L. Wilmer.

3313. [1928, 158] *Proposed by L. S. Johnston, University of Detroit.*

Consider the infinite sequence $[a_n]$ of real positive numbers with the recurrent relation $a^2_{k+1}=2a_k/(1+a_k)$ $(k=1, 2, 3, \cdots)$.

(1) Prove $L_{k\to\infty}a_k=1$ for every a_1.

(2) Prove $L_{n\to\infty}\prod_{k=1}^{n}a_k$ exists and is different from zero for every a_1.

(3) Express the limit in (2) as a function of a_1.

The problem is of course trivial for $a_1=1$.

Solution by the Proposer.

If $a_1=1$ it is evident that $a_k=1$ for all k's, and that $L_{n\to\infty}\prod_{k=1}^{n}a_k=1$.

Consider next the case for which $a_1>1$ and set $a_1=\sec\omega$, $0<\omega<\pi/2$. Then from the identity

$$\sec^2(\omega/2)=\frac{2\sec\omega}{\sec\omega+1},$$

we may set $a_2=\sec(\omega/2)$. Repeating this reasoning we obtain $a_n=\sec(2^{1-n}\omega)$. Hence it follows that a_n approaches the limit unity, and it decreases to this limit except in the trivial case $a_1=1$. We may now set

$$1\Big/\prod_{k=1}^{n}a_k=\cos\omega\cos 2^{-1}\omega\cos 2^{-2}\omega\cdots\cos 2^{1-n}\omega$$

$$=\frac{\sin 2\omega}{2^n\sin 2^{1-n}\omega}=\frac{\sin 2\omega}{2\omega}\,\frac{2^{1-n}\omega}{\sin 2^{1-n}\omega},$$

where the second form results by the transformation of each factor by means of the formula $\cos A=(\sin 2A)/(2\sin A)$. Hence we have

$$\lim_{n\to\infty}\prod_{k=1}^{n}a_k=\frac{2\omega}{\sin 2\omega}=\frac{a_1^2\sec^{-1}a_1}{(a_1^2-1)^{1/2}}.$$

For the case in which $a_1<1$ we may set $a_1=\operatorname{sech}\omega$, $\omega>0$. We have merely to replace the trigonometric formulae by the corresponding hyperbolic formulae, and the reasoning follows in a similar manner. We thus find that a_n approaches unity as a limit and it increases toward this limit, while

$$\lim_{n\to\infty}\prod_{k=1}^{n}a_k=\frac{a_1^2\operatorname{sech}^{-1}a_1}{(1-a_1^2)^{1/2}}.$$

Also solved by F. L. Wilmer.

3314 [1928, 154]. *Proposed by J. B. Reynolds, Lehigh University.*

A uniform flexible chain of length a, weight w, slides down an arch of a smooth inverted cycloid of total length $8a$ in a vertical plane, with its vertex lowest. If the chain is released from rest in a position on the arch in which one end is at one end of the arch, find the time until the middle of the chain is at

the lowest point of the cycloid, the speed of the chain at that instant and the tension at its middle point then.

I. *Solution by Wm. M. Borgman, Jr., College of the City of Detroit.*

Let the equations of the inverted cycloid be written in the form, $x=a(\theta-\sin\theta)$, $y=a(1+\cos\theta)$. Then the length of arc, measured from $(0, 2a)$, is $s=4a(1-\cos\frac{1}{2}\theta)$.

Let $w=$weight per unit length of the chain, $K=$kinetic energy of the chain and $P=$potential energy when one end of the chain is at the point $s=s$ and the x-axis is the datum line. Then

$$K = wav^2/2g, \quad P = w\int_s^{s+a} y\,ds = -\frac{8}{3}a^2w\cos^3\frac{\theta}{2}\Big]_s^{s+a};$$

and

$$\frac{dP}{ds} = 4a^2w\cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\cdot\frac{d\theta}{ds}\Big]_s^{s+a} = 2aw\cos^2\frac{\theta}{2}\Big]_s^{s+a} = \frac{w}{8}(2s-7a),$$

since $ds/d\theta=2a\sin\frac{1}{2}\theta$. From the known fact that $dK/dt=-dP/dt$, we have $(wavg^{-1})dv/dt=-(dP/ds)v$, and hence $8a\ dv/dt=(7a-2s)g$, or $(d^2s/dt^2)+(g/4a)s=(7g/8)$.

The solution is

$$s = c_1\cos(g/4a)^{1/2}t + c_2\sin(g/4a)^{1/2}t + 7a/2.$$

When $t=0$, $s=0$, $v=0$. Hence

$$s=(7a/2)[1-\cos(g/4a)^{1/2}t] \text{ and } v=(7a/4)(ag)^{1/2}\sin(g/4a)^{1/2}t.$$

When the middle of the chain is at the vertex, $s=7a/2$, $t=\pi(a/g)^{1/2}$, and $v=(7/4)(ag)^{1/2}$.

Since the chain is moving with uniform velocity when the center is at the vertex of the cycloid, the chain may be considered at rest for the purpose of calculation of forces along its length.

Let T be the tension in the chain when its middle point is at the vertex of the cycloid and let ϕ be the angle between the horizontal and the tangent.

Then $dT=w\sin\phi\,ds=-aw\sin\theta d\theta$, since $\sin\phi=-\cos\frac{1}{2}\theta$ and $ds=2a\sin\frac{1}{2}\theta\,d\theta$. Hence $T=2aw\cos^2(\frac{1}{2}\theta)-(aw/32)$, where the constant of integration is chosen so that $T=0$ when $s=9a/2$.

Hence the tension at the middle of the chain (i.e., where $\theta=\pi$) is $-wa/32$.

II. *Solution by the Proposer.*

Let s measure the arc distance from the vertex of the cycloid to a variable point P on the chain and let z measure the arc distance from the vertex of the cycloid to the left end of the chain assuming that the right end is initially at the right cusp of the arch. Let T be the tension in the chain at the point P, ϕ the angle the normal at P makes with the vertical, and $v=dz/dt$ the speed of the chain.

Resolving forces along the tangent at P we find the equation

$$dT/ds = (W/a)\sin\phi + (W/ga)vdv/dz, \tag{1}$$

in which T, s, and ϕ may vary independently of z and v. Now $s=4a\sin\phi$, and therefore we get from (1), upon integrating,

$$T = 2W\sin^2\phi + (4W/g)(vdv/dz)\sin\phi + C.$$

Now $T=0$ at the left end of the chain, so we have $T=0$ when $s=z=4a\sin\phi$; whence, determining the constant C, we have

$$T = 2W\sin^2\phi - Wz^2/8a^2 + (4W/g)[\sin\phi - z/4a]vdv/dz. \tag{2}$$

Again $T=0$ at the right end of the chain where $s=z+a=4a\sin\phi$; whence by (2) we get

$$vdv/dz = -(g/8a)(2z + a). \tag{3}$$

Integrating and determining the constant of integration by the condition that $v=0$ for $z=3a$, we find

$$v^2 = (g/4a)[12a^2 - az - z^2] = (g/4a)(3a - z)(4a + z). \tag{4}$$

Equation (4) shows that the chain slides to the left until $z=-4a$, that is until the left end just reaches the left cusp. It therefore oscillates back and forth between the two cusps of the arch. The middle of the chain is at the lowest point when $z=-a/2$, and then $v=(7/4)(ag)^{1/2}$.

From equation (4) we have

$$dz/(12a^2 - az - z^2)^{1/2} = -\tfrac{1}{2}(g/a)^{1/2}dt,$$

and, upon integrating and noting that $t=0$ for $z=3a$, we have

$$t = 2(a/g)^{1/2}\arccos[(2z + a)/7a]. \tag{5}$$

When $z=-a/2$, $t=\pi(a/g)^{1/2}$; that is the chain reaches the lowest position in the same time as a particle starting from any point of the curve requires to reach the vertex.

Substituting v from (4) in (2) we find

$$T = (W/8a^2)(s - z)(s - z - a).$$

Where $s=z+\frac{1}{2}a$ (the middle point of the chain for any position), we find $T=-W/32$; that is, in all positions the chain at its middle point is under a compression equal to one thirty-second of its weight. Furthermore, since $(s-z)$ is the distance of P from the left end of the chain, we see that the chain at all points except its ends is under a compression which remains constant throughout the motion.

Also solved by William Hoover.

3315 [1928, 207]. *Proposed by Harry Langman, Arverne, N. Y.*

Show that

$$x^2 + y^2 + z^2 - yz - zx - xy + u^2 + v^2 + w^2 - vw - wu - uv \geqq 3^{1/2} \cdot \begin{vmatrix} u & x & 1 \\ v & y & 1 \\ w & z & 1 \end{vmatrix}$$

for all real values of the variables. Also find the conditions under which the equality obtains.

Solution by Frank Ayres, Jr., College Station, Texas

Let $A=(u, x)$, $B=(v, y)$, $C=(w, z)$ be three points determining a triangle for which sides will be denoted by a, b and c and area by D. Then, we are to show that

$$(a^2 + b^2 + c^2)3^{1/2}/4 \geqq 3D.$$

Of the triangles having a given perimeter $a+b+c=3p$, the equilateral triangle has the maximum area, $3^{1/2}p^2/4$. Also,[1] $a^2+b^2+c^2>3p^2$, unless $a=b=c=p$, in which case the two members are equal.

Hence the equality obtains when the triangle ABC is equilateral; otherwise, the inequality holds.

3317 [1928, 207]. *Proposed by Emma M. Gibson, High School, Springfield, Missouri.*

The coordinates of a point are expressed in terms of two parameters, α, β, by

$$x/a = (\alpha + \beta)/(\alpha - \beta), \quad y/b = (1 - \alpha\beta)/(\alpha - \beta), \quad z/c = (1 + \alpha\beta)/(\alpha - \beta),$$

where a, b, and c are constants.

Determine the locus of the point and determine the relations of the lines $\alpha=$constant and $\beta=$constant to the surface. Also show how to express the differential equation of the lines of curvature in terms of α and β.

Solution by J. H. Neelley, Carnegie Institute of Technology.

The locus of the point is obviously the hyperboloid of one sheet,

$$a^{-2}x^2 + b^{-2}y^2 - c^{-2}z^2 = 1 \tag{1}$$

The lines $\alpha=$constant and $\beta=$constant are the two systems of rectilinear generators of the surface with rectangular equations

$$\begin{aligned} b^{-1}y - c^{-1}x - \alpha(1 - a^{-1}x) &= 0, \\ 1 + a^{-1}x - \alpha(b^{-1}y + c^{-1}z) &= 0; \end{aligned} \tag{2}$$

and

$$\begin{aligned} b^{-1}y - c^{-1}z + \beta(1 + a^{-1}x) &= 0, \\ 1 - a^{-1}x + \beta(b^{-1}y + c^{-1}z) &= 0. \end{aligned} \tag{3}$$

[1] Note by the Editors: See Theorem IV on p. 383 of vol. 34 (1927) of this Monthly.

The differential equation of the lines of curvature of the surface expressed in terms of α and β may be written

$$\begin{vmatrix} d\beta^2 & -d\alpha d\beta & d\alpha^2 \\ E & F & G \\ D & D' & D'' \end{vmatrix} = 0, \tag{4}$$

where E, F, and G are the coefficients of $d\alpha^2$, $d\alpha d\beta$, and $d\beta^2$ respectively in the metric of the surface, and D, D', and D'' are the coefficients of the same expressions in the second fundamental quadratic differential form of the surface. Equation (4) is derived in any treatise on Differential Geometry. See Art. 50 of Eisenhart's text.

Also solved by H. L. Dorwart.

Note by Otto Dunkel. The differential equation of the lines of curvature may be expressed in another form, and, after some computation, it becomes

$$[4a^2\alpha^2 + b^2(\alpha^2 - 1)^2 + c^2(\alpha^2 + 1)^2]d\beta^2 - [4a^2\beta^2 + b^2(\beta^2 - 1)^2 + c^2(\beta^2 + 1)^2]d\alpha^2 = 0.$$

It will be found that $D = D'' = 0$, so that the equation (4) in the above solution becomes $Gd\beta^2 - Ed\alpha^2 = 0$, and this equation in turn reduces to the one above.

3318 [1928, 208]. *Proposed by C. N. Mills, Normal, Ill.*

For the parabola, $y^2 = -2p(x-k)$, find the length (in terms of p and k) of the shortest segment of a line tangent to the curve included between the coordinate axes.

Solution by Watson M. Davis, Albion College.

For every value of m, the line $2m^2(x-k) - 2my - p = 0$ is tangent to the parabola. Its x- and y-intercepts are, respectively, $(2m^2k+p)/2m^2$ and $-(2m^2k+p)/2m$.

Let D be the length of the segment of this line included between the axes. Then

$$D^2 = (2m^2k + p)^2(1 + m^2)/4m^4.$$

We may consider p and k as positive and the independent variable as m^2, varying from 0 to ∞. The derivative of D^2 with respect to m^2 consists of one factor which is never negative and another factor, $2km^4 - pm^2 - 2p$, which varies from $-$ to $+$. Hence there is a minimum given by the positive root m^2 of this factor or by

$$4km^2 = p + (p^2 + 16kp)^{1/2}.$$

This value of m^2 inserted in the above expression for D gives the minimum required.

Also solved by Frank Ayers, Jr., Nina M. Alderton, W. J. Patterson, E. G. Olds, and Fredrick Woods.

NOTES AND NEWS

Readers are invited to contribute to the general interest of this department by sending items to H. W. Kbhn, Ohio State University, Columbus, Ohio.

On November 18, 1928, occurred the death of Alexander Ziwet, Professor Emeritus of Mathematics at the University of Michigan, one of the outstanding men in mathematics in this country during the last forty years.

The principal land marks of his life were as follows: born, February 8, 1853 at Breslau, Germany; graduated from the gymnasium, 1870; attended the University of Warsaw, 1871–3 and the University of Moscow, 1873–4; attended the Polytechnische Hochschule in Karlsruhe, 1876–1880; employed by the Lake Survey and the U. S. Coast and Geodetic Survey at Detroit, 1880–8; appointed instructor, then assistant professor, associate professor, professor, and finally head of the department of mathematics in the College of Engineering, University of Michigan, 1888–1925.

There was recognition of his ability in his being Associate Editor of the Bulletin of the American Mathematical Society, 1892–1920; Vice President of the Society, 1903; Vice President and Chairman of Section A, American Association for the Advancement of Science, 1905–6. He was a member of the first executive board of the Mathematical Association, and a member of its committee on relations with the Annals of Mathematics. The University of Michigan conferred on him the honorary degree of Sc.D. in 1927.

Professor Ziwet was outstanding as a scholar and a teacher. His range of knowledge was not limited to mathematics, especially from the applied point of view, but extended to many sections of pure mathematics, history of mathematics and the humanities. As a linguist he was perhaps unsurpassed by any member of the University faculty. As a teacher and lecturer especially during his prime he was unexcelled. Many an Engineering College graduate looks back with pride on the fact that he was taught his theoretical mechanics by Professor Ziwet, and many a graduate student and colleague was delighted and inspired by the clarity, accuracy, and balance of his presentations. He was a potent influence in the University, not only for high ideals in connection with engineering education, but also in the promotion of graduate work and research. His ideas on such matters were often far in advance of those of his colleagues and of the time. As an individual he was a gentleman in the fullest sense of the word, honest, just, upright, generous, faithful, modest, and retiring.

The UNIVERSITY *of* WISCONSIN

Summer Session 1929

General Session, **July 1 to August 9. Fees: $24.50. Graduate School, $33.50.**

Special Graduate Courses, **July 1 to August 30. Fee: $48.50.**

Law School, **June 24 to August 30. Fee: $38.50.**

Full program of courses in undergraduate and graduate mathematics. Special attention given to courses in the teaching of mathematics. Fine library and equipment for the use of students wishing to work for higher degrees.

Nine-Weeks Courses: Advanced electrodynamics; theory of analytic functions; higher algebra; theory of relativity; theory of approximations.

Favorable Climate Lakeside Advantages

For Literature, address

DIRECTOR, SUMMER SESSION
Madison, Wisconsin

Missing Numbers of the Monthly

Cash, or credit toward future dues, will be given for certain single numbers as follows, up to a limited number of copies: February, March, May, September, 1913; September, 1914; February, March, April, June, 1915; February, September, 1918—fifty cents; September, 1915—seventy-five cents; May, 1915—one dollar. (See MONTHLY, March, 1921, p. 152); October, 1920; August-September, October, 1921; May, September, October, 1924; May, June-July, November, 1926 — forty-five cents.

Address all communications to the
Secretary, W. D. Cairns
Oberlin, Ohio

Legal Form for Gifts and Bequests

I hereby give[1] to the Board of Trustees of the Mathematical Association of America the sum of ..... Dollars, to be known as the ..... Fund, and to be used for[2] { Endowment—the income only of which may be expended. / Special Projects—for which both principal and income may be expended. }

Witness: ..... Signature .....

[1] In case of a bequest, the first line should read "I hereby give and bequeath," etc.
[2] Indicate which one of the two purposes is desired, and omit the other.

The Association needs funds for scientific publications and for the promotion of scientific activities.

CONTENTS

The Information Bureau for Appointments 179
The Fundamental Mathematical Requirements of Biology. By J. ARTHUR HARRIS 179
On Certain Sequences of Conics and Associated Sequences of Numbers. By L. S. JOHNSTON 198
A Note on Newton's Diagram for Approximating Plane Algebraic Curves and Surfaces. By ALAN D. CAMPBELL 204
Linear Fractional Transformations on Quartic Equations. By RAYMOND GARVER 208
QUESTIONS AND DISCUSSIONS: Discussions—"Standard Deviation Problems with Rational Answers," by WALTER CROSBY EELLS; "Marginal Notes," by T. H. HILDEBRANDT; "The Cheney Check Formulas," by WILLIAM R. RANSOM; "A Letter to the Editor," by LEONARD M. BLUMENTHAL 212
RECENT PUBLICATIONS: Reviews by R. C. ARCHIBALD, ARTHUR B. COBLE, R. D. CARMICHAEL, GEORGE A. HARTER, R P. STEPHENS, C. A. RUPP, R. D. CARMICHAEL, LAO G. SIMONS 222
PROBLEMS AND SOLUTIONS: Problems for Solution—3372–3375. Solutions —3310, 3311, 3313, 3314, 3315, 3317, 3318 232
NOTES AND NEWS 240

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. H. BUSSEY, 106 Folwell Hall, University of Minnesota, Minneapolis, Minn.

BOOKS FOR REVIEW should be sent to R. A. JOHNSON, Hunter College, New York, N. Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, Oberlin, Ohio.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Thirteenth Summer Meeting of the Association, Boulder, Colorado, August 26-29, 1929.
Fourteenth Annual Meeting, Des Moines, Iowa, December 31, 1929, January 1, 1930.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled.

ILLINOIS, Carthage, Ill., May 3-4.
INDIANA, Culver Military Academy, May 3-4.
IOWA, Fairfield, Iowa, April 26-27.
KANSAS, Topeka, Kansas, February 2.
KENTUCKY, Lexington, Ky., April 13.
LOUISIANA-MISSISSIPPI.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, George Washington University, May 4.
MICHIGAN, Ann Arbor, Mich., March 16.
MINNESOTA, St. Paul, May 8.
MISSOURI, Kansas City, Mo., November.
NEBRASKA.
OHIO, Columbus, Ohio, April 4.
PHILADELPHIA, University of Pennsylvania, November 30.
ROCKY MOUNTAIN, Greeley, Colo., April 12-13.
SOUTHEASTERN, Macon, Ga., April 19-20.
SOUTHERN CALIFORNIA, University of Redlands, March 9.
TEXAS, Houston, Texas, Jan. 26.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

THE INFORMATION BUREAU FOR APPOINTMENTS

Members of the Association are reminded that the Association maintains an office for supplying information with regard to men and women available for appointment to college positions in mathematics. This office does not handle detailed recommendations, after the manner of a teachers' agency, but supplies certain essential facts with regard to each candidate, together with the name of a sponsor from whom further information about him can be obtained. The aim is to keep the files as complete and up-to-date as possible. To this end, candidates for appointment, especially candidates for a first appointment, are invited to put their names on record with the office, and departments in search of instructors are urged to avail themselves of its facilities. There is no charge for its services, either to departments or to candidates. Registration blanks and information may be obtained from Professor H. W. Kuhn, Ohio State University, Columbus, Ohio.

THE DECEMBER MEETING OF THE MARYLAND-VIRGINIA-DISTRICT OF COLUMBIA SECTION

The twenty-fourth regular meeting of the Maryland-Virginia-District of Columbia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Md., on Saturday, December 8, 1928. Sessions were held in the morning and in the afternoon; Professor C. C. Bramble, chairman of the Section, presided at both sessions.

Forty-six persons attended the meeting, including the following thirty-three members of the Association: O. S. Adams, R. N. Ashmun, H. G. Avers, Clara L. Bacon, W. J. Berry, G. A. Bingley, L. M. Blumenthal, C. C. Bramble, G. R. Clements, A. Cohen, Tobias Dantzig, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, Harry Gwinner, L. S. Hulburt, F. E. Johnston, H. P. Kaufman, L. M. Kells, W. D. Lambert, Florence P. Lewis, A. K. Mitchell, Frank Morley, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, Jr., W. F. Reynolds, A. W. Richeson, R. E. Root, John Tyler, Paul Wernicke, E. W. Woolard.

During the intermission between the morning and the afternoon sessions, those attending the meeting were entertained at luncheon by the University; a hearty vote of thanks was passed in appreciation of the generous hospitality shown by the University. Mr. Edgar W. Woolard, acting secretary for the meeting in place of Rev. E. C. Phillips, S.J., whose transfer to New York had necessitated his resignation, was appointed permanent secretary of the Section, and was requested to convey to Dr. Phillips the congratulations of the Section on his transfer to a larger field of usefulness, their appreciation of his past services as Secretary, and their regrets that he could no longer be with them.